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ELEMENTS

OF

MATHEMATICS.

CONTAINING

GEOMETRY.	MECHANICS.
CONIC-SECTIONS.	PROJECTILES.
TRIGONOMETRY.	HYDROSTATICS.
SURVEYING.	HYDRAULICS.
LEVELLING.	PNEUMATICS.
MENSURATION.	*A* THEORY *of* PUMPS.
LAWS *of* MOTION.	

To which are prefix'd,

The FIRST PRINCIPLES of *ALGEBRA*,

BY WAY OF INTRODUCTION.

For the Use of the Royal Academy *of Artillery at* WOOLWICH.

VOL. I.

By JOHN MULLER,

Profeſſor of *Artillery* and *Fortification.*

LONDON:

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TO

Martin Folks, Esq;

SIR,

AS I am under the greatest Obligations to You, in many Respects, I should, with Pleasure, take this Opportunity to testify my grateful Sense of Your Favours, were it in my Power to do it without being suspected of the Flattery so peculiar to Dedications.

But as I ever looked upon Ingratitude with the utmoſt Abhorrence, I cannot help addreſſing myſelf to You in this public Manner, hoping, Sir, you will be ſo kind as to conſider my Inability of making you any other Return; and believe that no Man was ever more ſenſibly affected than myſelf with Your unbounded Generoſity; and that I ſhall always be, with the greateſt Truth and Reſpect,

SIR,

Your moſt humble,

Obliged, and

Obedient Servant,

Royal Academy at Woolwich. JOHN MULLER.

PREFACE.

IT may ſeem ſuperfluous to write the Elements of a Science which has been ſo much cultivated in all Ages, as the Mathematics, and brought to ſuch Perfection by the incomparable Sir *Iſaac Newton*, and ſome other eminent Men cotemporary with him, that it may be queſtioned whether there be any room left for farther Improvement: Nevertheleſs, if it be impartially conſidered, it will be found, that though there are ſeveral very excellent Authors, with regard to the ſpeculative Part; yet thoſe who have endeavoured to render this ſublime Science uſeful, by a proper Application of its Principles, to the Practice and Perfection of mechanical Operations, are very few.

The Royal Academy of Artillery at *Woolwich* being intended for the Inſtruction of young Gentlemen, in whatever may be uſeful and conducive to the Improvement of Military Science, and the ſeveral Parts contained in theſe Elements being there taught and practiſed, we imagined that the publiſhing of them might be acceptable, not only to thoſe who attend the Academy, but likewiſe to others who are deſirous of being informed of theſe uſeful Branches of the Mathematics.

Plane Geometry being the Foundation of many Arts and Sciences, it is of the greateſt Importance to treat it with ſuch Perſpicuity, that a Scholar of a moderate Genius, with due Application, may in a ſhort Time acquire a competent Knowledge therein: This has been our chief Aim; how far we have ſucceeded, muſt be ſubmitted to the Judgment of the Reader. We have followed *Euclid* as nearly as we could, with reſpect both to the Manner and Strictneſs of Demonſtration, too much neglected by modern Authors, who, under Pretence of Brevity, change the geometrical Purity and Rigour of the Antients, for Algebraic Computations, whereby this noble Science, which ſhould be perfectly clear and independent, is obſcured, and rendered dependent on foreign Principles.

We have, it is true, differed ſomewhat from *Euclid*, with reſpect to the Order of the Propoſitions and Books; becauſe we could not otherwiſe have reduced it into ſo ſmall a Compaſs as we have done; and it ſeemed indifferent to us what Order was followed, provided that the Evidence of the Propoſitions always depended on, and had a Connection with, what had been antecedently demonſtrated.

The Doctrine of Proportions being the principal and moſt difficult Part of Geometry, we have endeavoured to explain it in the cleareſt Manner we could think of, and to deduce it from ſuch Principles as appeared natural, and at the ſame Time moſt intelligible. Moſt of the Definitions given by *Euclid* appear obſcure and perplexed to moſt Readers; and thoſe given by others being imperfect, we have choſen a Medium between both, whereby we preſume to have avoided the In-

Inaccuracy of the one, and the Obſcurity of the other: Though the Poſtulatum of Multiples aſſumed by *Euclid*, is more ſimple than ours of equal Parts; yet our Demonſtrations, which are no leſs general than his, hereby become both ſhorter and plainer, as will appear to the impartial Reader by the Peruſal of both.

From the ſame Principles we have demonſtrated that important Proportion between the Squares of the Diameters and Areas of Circles, which Authors take ſo much Pains to prove in another manner. This they would certainly not have done, had they rightly underſtood *Euclid*'s fifth Definition; for it is manifeſtly general, and extends to all Kinds of Magnitudes, whether commenſurable or incommenſurable; and conſequently, whatever falls under the Conſideration of Proportions, may be deduced therefrom, without the Aſſiſtance of any foreign Principles.

Euclid's Definition of Compound Ratios has been variouſly interpreted by the Tranſlators. Dr *Barrow*'s is an Aſſumption too arbitrary to be admitted for a good Definition, neither is it agreeable to the 23d Propoſition of the ſixth Book, no more than thoſe given by Dr. *Gregory* and *Keil*, of which the latter was ſo ſenſible, that he premiſes a Lemma to this Propoſition, but unaccountably demonſtrates this Lemma from thoſe very Principles which he condemns at the Beginning of the fifth Book: There is a Prolixity in the Demonſtration of this Propoſition, which does not ſeem to agree with *Euclid*'s Manner of writing; for there is no manner of Occaſion to aſſume Lines, ſince it may be done much eaſier without them, by Means of our Definition.

Notwithſtanding the great Pains we have taken to render this Subject plain and eaſy, yet ſuch is the Nature of it, that the firſt Part, which treats of Magnitudes in general, will not, I fear, be readily underſtood by Beginners, who are therefore deſired to paſs it over after they have read the Definitions, and proceed to the ſecond, where all that had been demonſtrated before of Magnitudes in general, is in one Corollary, proved of Lines.

As it appears more natural to treat every Part of Geometry ſeparately, the firſt Section contains what is neceſſary to be known of Lines and Angles; the ſecond of Triangles; the third of Parallelograms; and ſo on: In the ſame Manner we have given the moſt neceſſary Problems in a ſeparate Section. Hence all the Propoſitions are preſented together to the Reader's View, who may afterwards read as many Problems as he thinks proper, leaving the others till ſuch Time as they are wanted. The Circle has been treated of after the Doctrine of Proportions, becauſe its Properties are from thence demonſtrated in a very ſhort Manner; and that with reſpect to the Meaſure of Angles is deduced from thoſe to which it has an immediate Connection, which could not have been done otherwiſe.

Having given in the firſt eight Sections every Thing we conceived to be uſeful and neceſſary, of Plane and Solid Geometry, the ninth contains a new Method of inveſtigating the Areas of curvilinear Spaces, and the Contents of Solids, in a general Manner, depending only on the Knowledge of Plane Geometry and common Algebra, illuſtrated with common Examples, ſuch as the Length of an Arch, the Area of a Circle, the

the ſolid Content of a Sphere, and other regular Bodies; and as this is done without Fluxions, to which this Inveſtigation is a kind of Introduction, it is hoped that the Reader will be pleaſed to ſee this general Method deduced from ſuch eaſy Principles.

The ſecond Book, which treats of Conic-Sections, contains the moſt noted Properties only, and ſuch others which are neceſſary in the enſuing Part of this Work; and as Logarithms are very uſeful in Computations, we have deduced their Conſtruction from the Properties of the Hyperbola, in which we have followed Sir *Iſaac Newton*, preferably to thoſe who derive it from Numbers, merely becauſe they are obliged to have Recourſe to an infinite Series, the Demonſtration of which depends on Fluxions; and to ſuppoſe a ſmall Number to be infinite, whereby their Inveſtigation becomes ſo intricate and obſcure, that few Readers are able to underſtand it.

As the two preceding Books contain the elementary Part of Geometry, ſo the two ſucceeding ones comprehend its moſt uſeful Applications to the different Practices, chiefly neceſſary to an Engineer: For Trigonometry, Menſuration, and Surveying, are not only applied to common Uſes, but likewiſe to Fortification, ſo far as is neceſſary for finding of Lines and Angles, as alſo the Quantity of Maſonry in ſtrait Walls, concave Flanks, Orillons, Vaults, and Arches of Powder Magazines, and ſuch-like Buildings.

In Mechanics we have endeavoured to demonſtrate all the principal Propoſitions in a plain and familiar Manner; the Properties and Uſes of the five ſimple Inſtruments have been conſidered ſe-

perately, and applied to ſome ſimple Machines, by the Help of which, thoſe that are more compound, and could not be inſerted here, with a proper Application of what has been ſaid before, may be readily comprehended. To render the Principles of this Science more conſpicuous, they have been applied to the Preſſures of Earth againſt Walls, and to the proper Thickneſſes of Walls to reſiſt this Preſſure; this is done in a much ſhorter Manner than what we meet with on this Subject in ſome *French* Authors: For our Equations are ſo ſimple, that the proper Thickneſs of a Wall is found by only multiplying its Height by a given Number, ſo that there is no Occaſion for Tables.

We have alſo conſidered the beſt Manner of making Wheel Carriages, a Subject very uſeful in Artillery, in order to determine the Figure of the Axletrees, ſo as the Wheels may cauſe as little Friction as poſſible, whereby the Draught of ſuch heavy Carriages is conſiderably leſſened. Dr. *Deſaguliers* has taken Notice of the ſame Thing at the End of his Experimental Philoſophy; but as we had not ſeen it before our Diſcourſe was printed, his Name was not mentioned.

As Authors who have written on Projectiles, only conſidered them in a non-reſiſting Medium, we, after having given the common practical Rules, have inſerted the moſt uſeful Problems relating to Projectiles deſcribed in a Medium, whoſe Reſiſtance is as the Square of the Velocity, and ſhewn the Manner of making proper Experiments, in order to determine ſuch Quantities as are neceſſary to be known before this Theory can be applied to Practice: But becauſe the Demonſtrations depend on the Method of Fluxions, they

are

are omitted; thoſe who are deſirous to know the Grounds on which they depend, may conſult *Euler*'s Mechanics, who is the only Author I know, that has conſidered this Kind of Projectiles.

Conſidering the Uſefulneſs of the Method for finding the greateſt and leaſt Quantities; and that it is not ſo much applied to Practice as it ſhould be, on account of its Principles depending on Fluxions, which few practical People underſtand, what we have delivered on that Subject, is demonſtrated by common Geometry: It conſiſts in finding the greateſt Rectangle that can be inſcribed in a given Segment of a Conic-Section; and its Demonſtration depends on this obvious Propoſition, that the Complements of a Parallelogram are equal: Though we have proſecuted the Subject no farther than was neceſſary to ſolve the mechanical Problems of this Kind, treated of in this Work, yet it may eaſily be perceived, that it extends to all Caſes in which the Solution depends on a ſimple or quadratic Equation.

In treating of Hydraulics we met with great Difficulties, inaſmuch as all the Authors we have ſeen on this Subject, diſagree in their Principles, with reſpect to Water running through Orifices; and notwithſtanding we have conſulted the moſt noted Writers, yet we could find but little that might be uſeful in ſettling the Principles of this Science on a ſolid Foundation; neither have we a ſufficient Number of Experiments to be depended upon, without which it is impoſſible to arrive at any degree of Certainty, with reſpect to this Subject.

As we differ from all thoſe Authors that have come to our Knowledge, it cannot be expected

that what we have said on this Subject should be entirely free from Objections; all we can offer in its Favour is, that it perfectly agrees with the few good Experiments extant, as also with those particular Cases which may be demonstrated from unexceptionable Principles, and in which most Authors agree; this, we presume, is not the Case of others.

For, according to Sir *Isaac Newton* and Mr. *Daniel Bernoulie*, when the Orifice is pretty large, the Velocity of the Water is very great, and infinite when the Orifice is equal to the Bottom of the Vessel; whereas it is manifest, that Water can never acquire a greater Velocity, by the Force of Gravity only, than other Bodies falling through the same Space; and therefore what these two great Men have said on this Subject, cannot be true in general, whatever it may be in particular Cases.

The Theory of Pumps being of general Use, especially to Engineers, in the Building of Forts and Castles, where Water is a material Article in the Maintenance of a Garrison, we thought that the Principles of Hydraulics could not be better applied than to these Machines; and to give a clear Idea of them we have treated the Parts separately, and shewn their Use as well as their Constructions; as also the Manner of working the Pumps, and what their Effect will be when executed. In short, nothing has been omitted that we thought might any ways contribute to a thorough Knowledge of the Subject.

As Mr. *Belidor*, who has treated this Subject in a very ample Manner, has made several considerable Mistakes, some of which have been copied by Dr. *Desaguilers* in his Experimental Philoso-

phy,

phy, we presume that the correcting Part of them, and pointing out others, will be acceptable to those who apply themselves to that Study, especially as this Author is very good in all other Respects, and has given the best Treatise of Machines now extant.

I am perswaded that the Writing of Elements is looked upon by many as a very easy Thing; but if the great Nicety required to connect all the different Parts together, in such a Manner as to compose a continual Chain of Reasonings, deduced from the plainest Principles, and within Reach of most Capacities, be considered, it will be found a harder Task than is imagined. I do not doubt but this Work will meet with many Censures of opposite Kinds; those who justly look upon *Euclid* as the most correct Author, will blame me for having altered his Order and Disposition, as well as for having changed some of his Definitions, without considering the Time or the Reasons which induced him to be so very exact in his Reasonings; nor that it was then sufficient to understand the Elements of Geometry to be a complete Mathematician: Whereas now it is no more than the Beginning of Mathematical Studies; and therefore it would be Time ill spent, to dwell too long upon one Subject. Others, who look upon the Antients with less Veneration, will think this Part of our Work too tedious, and would have been better pleased to see it treated in a different Manner.

As our Intention is to instruct Youth in the most material Parts of the Mathematics, with respect both to Theory and Practice, in the plainest and shortest Manner, we shall submit our Endeavours to the Judgment of the impartial Reader,

and

and not take it amiſs if any Faults which may inadvertently have crept in, are taken notice of; but if it meets with inſignificant or trivial Cenſures, as is too often the Caſe, we ſhall not think it worth while to take any notice of them.

There has lately been publiſhed a Book under the Title of Elements of Plane Geometry, deſigned for the Uſe of Schools, which is an incorrect Copy of the firſt eight Sections of this Work, lent the pretended Author upon a particular Occaſion, and printed in a ſpurious Manner, without my Knowledge or Conſent; an Action too ſcandalous for any Man of Honour to be guilty of. The Editor imagined, I ſuppoſe, that the changing ſome Propoſitions, and mangling the Demonſtrations of others, was a ſufficient Diſguiſe to make it paſs for his own Performance; but how far this will juſtify ſuch a Piece of Pyracy, muſt be left to the Judgment of the Public.

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Euclidis Elementorum. *Oxon.* 1731. By Dr. *Keill.*
Euclidis Elementorum. *Oxon.* 1703. By Dr. *Gregory.*
Mr. *Belidor*'s Works, printed at *Paris.*
Mathematical Treatiſe. By the Author.
Surveying by *Hammond. Lond.* 1731.
Hydroſtatic Lectures by Dr *Cotes. Lond.* 1738.
Lectures of Natural Philoſophy, by Dr. *Helſham. Lon.* 1719.
Sir *Iſaac Newton*'s Principia. *Lond.* 1729.
Daniel Bernoullie's Hydrodynamicæ. *Argentorati* 1738.
Mr. *Mariotte*'s Works. *Leide,* 1717.

ERRATA.

PAGE 10, line 12, for ſides read ſide; p. 15, l. 3, beſies, r. beſides; l. 18, FF, r. DE; p. 27, l. 32, ÷A:B:C::D, r. ÷A:B:C:D; p. 44, l. 4, F r. E; p. 77, l. 8, 63 r. 60; p. 99, l. 3, 17, and 30, reciproque r. reciprocal; p. 100, l. 3, reciproque r. reciprocal; p. 113, l. 15, ++ r. +; p. 125, l. 11, $\frac{2+z^3-3zz+z}{2\times 3}$, r. $\frac{2z^3+8zz+z}{2\times 3}$; p. 137, l. ult. in the the marg. r. art. 235; p. 143, l. 9, $\overline{QN}$ r. $\overline{QN}^2$; p. 146, l. 19, $\overline{CA}$ r. $\overline{CA}^2$; p. 150, l. 14, $\overline{QR}$ r. $\overline{QR}^2$; p. 151, l. 5, MKm, r. MRm; p. 156, l. 11, their differences, r. the difference; p. 299, l. 29, ſuppoſition, r. ſuſpenſion; p. 314, l. 11, *af*, r. *ae*; p. 320, l. 4, degrees, r. inches; p. 324, l 15, CED, r. CBD; p. 324, l. antepenult. FCBA, r. FDBA; p. 325, l. antepenult. the W, r. the weight W; p. 333, l. 14, as their, r. as the ſquares of their; p. 344, l. 27, 28, DA*e* being added in fig. 3, or ſubtracted in fig. 4, r. DAF bing added in fig. 4, or ſubtracted in fig. 3; p. 346, l. 31, 25, r. 15; p. 373, l. 12, of one, r. one; p. 384, l. 12, mean, r. the mean; p. 400, l. 29, L, r. I; p. 409, l. 6, LD, r. LA; p. 410, l. 27, DH, r. LH; p. 417, l. 5, l. 10, r. 100.

Soon will be publiſhed, by the ſame Author,

A Compleat Treatiſe of ARTILLERY, containing the Proportions and Conſtructions of all Military Engines uſed here in England, and in France; with an Examination of their Perfections and Imperfections; as likewiſe general Conſtructions of Guns, Mortars, and their Carriages, applied to any Size of Guns or Mortars. Illuſtrated with a great many Copper-Plates.

IN-

INTRODUCTION.

THOUGH we have given ſintetical demonſtrations throughout this Work, wherever the ſubject would permit us, we could, however, not help to uſe algebraic ſymbols on ſeveral occaſions, without rendering ſome parts incompleat, and others not altogether ſo diſtinct as could be wiſhed; and as in elements the reader ought not to be referred to any other work, we thought it was proper to give, in a few words, a diſtinct notion of the firſt principles of algebra, ſo far as was neceſſary to underſtand the enſuing work, by way of introduction, for the ſake of thoſe who are unacquainted therewith.

Algebra, which is alſo called *Analyſis*, partakes both of vulgar arithmetic, and of geometry, inaſmuch that the common rules are arithmetical, and the application to the ſolutions of queſtions and problems is founded upon the ſame axioms as geometry; moreover, the expreſſions found thereby may ſerve either to deduce rules for obtaining certain numbers, required by a queſtion or theorems, with regard to the geometrical figures: The truths diſcovered by algebra are therefore no leſs clear and evident, than thoſe found by geometry, and often are more extenſive and general, which is its greateſt excellence.

The relation of equality is expreſſed by the ſign $=$, that of inequality by $>$; thus, to expreſs that a is equal to b, we write $a=b$, and if we would expreſs that a is greater than b, we write $a>b$, or that a is leſs than b, we write $a<b$.

As quantity may be conſidered to have parts, or is capable of being greater or leſs, it is increaſed by addition, and diminiſhed by ſubtraction, which therefore are the primary operations that relate to quantity: The ſign + *Plus* is the mark of addition, and the ſign — *Minus* that of ſubtraction; thus, when the two quantities repreſented by a and b are to be added, we write $a+b$, when b is to be ſubtracted from a, we write $a-b$; and when ſeveral quantities are connected together, as $a+b-c+d$; the ſigns ſerve to ſhew which are to be added and which are to be ſubtracted; thus, if $a=6$, $b=8$, $c=7$, $d=5$, then $a+b-c+d=6+8-7+5=12$; for 6, 8, and 5, are to be added, and 7 ſubtracted from their ſum 19.

It muſt be obſerved, that the ſign + or — belongs to the quantity to which it is prefixed; as in $a+b$, or $a-b$, the ſigns belong to b; and as all quantities are ſuppoſed to have either the ſign +, or —, a quantity which has none is always ſuppoſed to have the ſign +; thus a, or $+a$, is the ſame thing.

Quantities are ſaid to be poſitive when they have the ſign +, and negative, when they have the ſign —; thus a, or $+a$, is a poſitive quantity; and $-a$, a negative one. The ſigns + or —, not only expreſs addition and ſubtraction, but likewiſe oppoſite affections of quantities; as for example, money in poſſeſſion, and money owing; oppoſite directions in motion; lines drawn contrary ways; contrary actions of powers; as will be ſeen hereafter.

Though $+a$ and $-a$, are equal as to quantity; yet as they are different in their affections, we

we do not suppose in algebra that $+a = -a$, because to infer equality in this science, quantities must not only be equal as to magnitude, but likewise in their affections; so that in every operation they may have the same effect; for an excess may be equal to a defect, a debt to a real Stock, and yet they have in all operations a contrary effect.

The number prefixed to a letter is called *Coefficient*, and shews how often the quantity represented thereby is to be taken; thus in $2a$, the coefficient 2 shews, that a is to be taken twice; in $3a$, the coefficient 3, that a is to be taken thrice; and, in general, na, that a is to be taken n times; when a quantity has no coefficient, unity is understood to be its coefficient; thus a, b, or $1a$, $1b$, is the same thing.

As the product of any quantity multiplied by a number, may be multiplied again by another number; when several letters are wrote together, as ab, abc, or $3ab$, the meaning is, that the numbers represented by these letters are to be multiplied together, and when the letters are the same as aa, that the number a is multiplied by itself; and aaa, that the product aa is multiplied again by a; or, which is the same, aa represents the square of a, and aaa the cube.

The square aa of a, is sometimes wrote a^2, and the cube aaa is wrote a^3; the numbers 2, 3, are called the indexes or exponents, and shew that a^2 is the second power of a, and a^3 the third power of that number.

Quantities are said to be *Like*, when they are represented by the same letters. Thus $+4a$ and

$-5a$, or $3abc$, and $-abc$, are like; but a and b, or $3bc$ and $-5cd$, are unlike. Quantities are also said to be *Simple* when there is but one term, as $+ab$, $-cd$; and *Compound*, when there are several terms connected together by the signs $+$ and $-$, as $a+b-c+d$, or $3ab+5cd-7bd$.

ADDITION

Is performed by connecting the several quantities together with their proper signs.

To	$+a$	$+3a$	$2a+3c$
Add	$+3b$	$-5c$	$+4d$
Sum	$a+3b$	$3a-5c$	$2a+3c+4d$

To $4a+5b-7c$
Add $5x-7y+3z$

Sum $4a+5b-7c+5x-7y+3z$.

There are two cases in which addition may be shortened in the following manner.

CASE I.

If quantites are like, and have like signs, *add their coefficients together, and prefix the sum to one of the quantities with its sign.*

To	$+4a$	$-5b$	$a+b$
Add	$+3a$	$-7b$	$3a+4b$
Sum	$+7a$	$-12b$	$4a+5b$.

To $7ab-9cd+3ac-4ad$
Add $2ab-3cd+5ac-7ad$

Sum $9ab-12cd+8ac-11ad$.

This

This is evident, since $4a$ and $3a$ make $7a$; as also $-5b$ and $-7b$ make $-12b$.

In general, whatever the coefficient may be of like quantities, which have like signs, their sum will always express the number of times that the quantity is to be taken.

CASE II.

If like quantities have unlike signs, *subtract the lesser coefficient from the greater, and prefix the difference to one of the quantities with the sign of the greater.*

To	$+6a$	$-7b$	$-ab$
Add	$-2a$	$+3b$	$+3ab$
Sum	$+4a$	$-4b$	$+2ab.$

To	$4ab$	$-5cd$	$+6ac$	$-7ad$
Add	$3ab$	$+6cd$	$-9ac$	$+12ad$
Sum	$7ab$	$+cd$	$-3ac$	$+5ad.$

When several sums are to be added together, take the sum of all the positive terms, and that of all the negatives ones, and then add these two sums together as before.

	$+5a$	$-7b$
	$-3a$	$+4b$
	$-2a$	$+3b$
	$+4a$	$-5b$
Sum of the positive	$+9a$	$+7b$
Sum of the negative	$-5a$	$-12b$
Total	$+4a$	$-5b.$

$$\begin{array}{r} 3aa+bb-7cd+8ab \\ 2aa-2bb+4cd-3ab \\ aa+bb+cd-2ab \\ \hline 6aa \quad *-2cd+3ab. \end{array}$$

As there is here $+2bb$, and $-2bb$, to be added, it is plain that their ſum $2bb-2bb=o$; ſince by adding and ſubtracting the ſame quantity neither increaſes nor diminiſhes the total.

Note, that all like quantities are to be added together, though they are not placed directly under each other; for it is indifferent in what manner they ſtand, only for the ſake of order; like terms are generally placed under each other.

$$\begin{array}{r} ab-3cd+4ac \\ 5cd-6ab+3ac \\ \hline 2cd-5ab+7ac. \end{array}$$

When quantities are affected with like ſigns, the rule is evident from the notation and nature of ſigns; ſince one real ſtock may be added to another, and one debt to another debt; and when quantities have different ſigns, it is no leſs evident that they muſt be connected together therewith; ſince a real ſtock joined to a debt, or an exceſs to a defect, muſt have contrary ſigns: Therefore when the defect is leſs, equal to, or greater than the exceſs, the ſum will be poſitive, nothing, or negative; thus, if a man has 200 pounds, and owes 100, his worth is expreſſed by 200—100, or 100; if he has 300 pounds, and owes as much, it is expreſſed by 300—300=0; but if he has only 200, and owes 500, then he is ſaid to be rich by 200—500, or —300; that is, he muſt get 300 pounds before he is even with the world.

SUB-

SUBTRACTION

Is performed by changing the ſigns of the quantities to be ſubtracted into their contrary; that is, the ſign + into —, and — into +, then adding them together as in addition.

From	$+5a$	$-5b$	$+6ab$
Subt.	$+3a$	$-3b$	$-3ab$
Rem.	$+2a$	$-2b$	$+9ab.$

It is evident, that to ſubtract a negative quantity is to add its poſitive value; thus, $-3b$ ſubtracted from $-5b$ gives $-5b+3b=-2b$; and $-3ab$ ſubtracted from $+6ab$, gives $+6ab+3ab=9ab.$

From	$3aa-5ac+6ab$
Subt.	$3aa-6ac+3ab$
Rem.	$* + ac + 3ab.$

Here $3aa$ ſubtracted from $3aa$ leaves nothing, ſince $3aa-3aa=0.$

From	$13xx-15cx+17ac$
Subt.	$11xx+cx+14ac$
Rem.	$2xx-16cx+3ac.$

The proof of ſubtraction is addition, as in vulgar arithmetic.

MULTIPLICATION.

Regard muſt be had with reſpect both to the ſigns and the quantities.

RULE *for the* SIGNS.

If both factors have like signs, the product must be positive; and negative, if the signs are unlike.

RULE *for the* QUANTITIES.

If they are simple, *prefix the product of the coefficients to that of the quantities*; and if they are compound, *multiply the several parts of one factor, one after another, by all the parts of the other; then the sum of all these products will be the product required.*

Mult.	$+a$	$-2a$	$-6x$
by	$+b$	$+3b$	$-4a$
Prod.	$+ab$	$-6ab$	$+24ax.$

The operations of the signs is proved thus; when a positive quantity a is multiplied by a positive number n, it is plain that the quantity a is to be repeated as often as n contains units, and their product must be $+na$.

If a negative quantity $-a$ be multiplied by a positive number, then that quantity is to be taken as often as n contains units, and the product must be $-na$, since a negative quantity repeated ever so much, remains still a negative.

To multiply a positive quantity a by a negative number $-n$, it is plain that a is to be taken so many times negatively as n contains units; so that the product must be $-na$.

But if a negative quantity $-a$ be multiplied by a negative number $-n$, then $-a$ is to be taken as many times with a contrary sign, as n contains units; and to subtract a negative quantity is the same thing as to add its positive value.

Con-

Consequently $-a$, multiplied by $-n$, gives $+na$.

It must be observed, that in multiplication one of the factors is always a number; for a quantity may always be multiplied by a number, but money cannot be multiplied by money, nor a line by a line, surface, or solid. The similitude between a rectangle or solid, and a product, is explained in the third book, where we treat of mensuration.

Mult.	$4a-5x$	$7ab-9cd+xx$
by	$3b$	$4c$
Prod.	$12ab-15bx$	$28abc-36ccd+4cxx$.

When several letters are multiplied together, they may stand any how, only for distinction sake they are wrote according to their order in the alphabet, as ab, abc, $abcd$: For * $2\times3\times4$, $2\times4\times3$, or $4\times2\times3$, give the same product, 24. We begin always multiplication in algebra at the left, and go to the right; though it is indifferent which way it be done.

Mult.	$a+b$	$a-b$
by	$a+b$	$a-b$
	$aa+ab$	$aa-ab$
	$+ab+bb$	$-ab+bb$
Prod.	$aa+2ab+bb$	$aa-2ab+bb$.

For $a+b$ multiplied by a gives $aa+ab$, and $a+b$

* Note, *the sign* × *signifies multiplication*; *for* a × b, *or* ab, *is the same thing.*

$a+b$ by $+b$, gives $ab+bb$; these two products being added, gives $aa+2ab+bb$. Again, $a-b$ multiplied by a, gives $aa-ab$; and $a-b$ multiplied by $-b$, gives $-ab+bb$; the sum of these two products gives $aa-2ab+bb$.

Mult. $xx+xz+zz$ by $x-z$

$x^3+xxz+xzz$
$-xxz-xzz-z^3$

Prod. $x^3 - - * - - * - - z^3$

$a+b$ by $a-b$

$aa+ab$
$-ab-bb$

Prod. $aa - - * - bb.$

Mult. $2xx-3ax+4aa$ by $2x+3a$

$4x^3-6axx+8aax$
$+6axx-9aax+12a^3$

$4x^3 - - * - aax + 12a^3$

When different powers of the same quantity are to be multiplied together, the index of the product is always equal to the sum of the indexes of the factors; for $aa = a^2$, $aa \times aa = a^2 \times a^2 = a^4$, $2x \times 2x^2 = 4x^3$.

It is sometimes useful not actually to multiply compound quantities, but to set them down with the sign $\times$ of multiplication between them, drawing a line over each of the factors. Thus $\overline{a+b} \times \overline{a+b}$ expresses the product of $a+b$ multiplied by $a+b$.

The multiplication of compound quantities is proved as follows; to multiply $a-b$ by a, we have aa for the product of a multiplied by a; but as a is greater than $a-b$, by b, the product aa is too much by the product ab; and therefore $a-b$ mul-

multiplied by a, gives $aa - ab$; that is, $+a$ multiplied by $-b$ gives $-ab$.

And to multiply $a-b$ by $c-d$, we get $ac-bc$ for the product of $a-b$, multiplied by c; but c is greater than $c-d$ by d; and therefore the product $ac-bc$ is too much by the product $d \times a-b$, or $ad-bd$, which must consequently be subtracted from $ac-bc$, in order to get $ac-bc-ad+bd$ for the product of $a-b$ multiplied by $c-d$. Hence, $-d$ multiplied by $-b$, gives $+bd$.

DIVISION.

The same rule for the signs is observed as in multiplication; that is, *if both the divisor and the dividend have like signs, the quotient must be positive, and negative if they have contrary signs.*

RULE *for the* QUANTITIES.

I. If they are simple, *strike out the letters which are common to both divisor and dividend, divide the coefficients by their common measure, and set the remainder of the divisor, if there is any, under the remainder of the dividend, with a stroke between them.*

Dividend	ab	$+4ab$	$-6cd$	$+7abc$
Divisor	b	$+2b$	$-3b$	$-bd$
Quotient	a	$+2a$	$+2cd$	$-7ac$
			b	d

II. But when quantities are compound, *set the divisor to the left of the dividend, with a stroke between them, and divide one term after another of the*

the dividend, by the divisor, in the same manner as in common arithmetic.

$$2a \;)\; 6ab + 10ac \;(\; 3b + 5c$$
$$6ab + 10ac$$
$$0 \qquad 0$$

For $6ab$, divided by $2a$, gives $3b$ for the quotient; $3b$, multiplied by the divisor $2a$, gives $6ab$, which, being subtracted, leaves $+10ac$; and $10ac$, divided by $2a$, gives $5c$ for the second term of the quotient; the quotient $5c$, multiplied by the divisor $2a$, gives $10ac$; this product being subtracted from the remainder $10ac$, leaves nothing; so that $6ab + 10ac$, divided by $2a$, gives $3b + 5c$ for the quotient.

Division is proved by multiplication, as in vulgar arithmetic; for the quotient $3b + 5c$, multiplied by the divisor $2a$, gives the dividend $6ab + 10ac$.

Divisor $a + b$) $aa + 2ab + bb$ ($a + b$ Quotient.
$aa + ab$
$0 \quad + ab + bb$
$ab + bb$
$0 \quad 0$

As aa, divided by a, gives a for the quotient, and the quotient a, multiplied by the divisor $a + b$, gives $aa + ab$, which being subtracted from the dividend $aa + 2ab + bb$, leaves $ab + bb$; the first term ab being divided by a, gives $+b$ for the second term of the quotient; and b multiplied by the divisor $a + b$, gives $ab + bb$; this product being subtracted from the remainder $ab + bb$, leaves nothing; therefore $aa + 2ab + bb$ divided

vided by $a+b$, gives the quotient $a+b$. For $\overline{a+b} \times \overline{a+b} = aa+2ab+bb$.

Divisor $a-b$) $aa-2ab+bb$ ($a-b$ Quotient

$aa-ab$

$\circ \quad -ab+bb$

$-ab+bb$

$\circ \quad \circ$

If any term arising from the product of the quotient, multiplied by the divisor, cannot be actually subtracted from the remainder, it is set as a part of the remainder, with a contrary sign.

Divisor $a-b$) $aa-bb$ ($a+b$ Quotient.

$aa-ab$

$\circ \quad +ab-bb$

$ab-bb$

$\circ \quad \circ$

As the term $-ab$, of the product $aa-ab$, of the quotient a, multiplied by the divisor $a-b$, cannot be actually subtracted, it is set as a part of the remainder, with a positive sign.

Divisor $x-z$) $bxx-bzz$ ($bx+bz$ Quotient

$bxx-bxz$

$\circ \quad +bxz-bzz$

$+bxz-bzz$

$\circ \quad \circ$

Divisor $x-z$) cx^3-cz^3 ($cxx+cxz+czz$ Quotient

cx^3-cxxz

$\circ \quad +cxxz-cz^3$

$+cxxz-cxzz$

$\circ \quad +cxzz-cz^3$

$+cxzz-cz^3$

$\circ \quad \circ$

Here

Here the term $-cxxz$, of the first product, cannot be actually subtracted, so it is placed as a part of the remainder, with a positive sign; and the term $-xzz$, of the second product, is, for the same reason, set to the remainder, with a contrary sign.

$$\begin{array}{l} cxx+cxz+czz \\ x-z \\ \hline \end{array}$$

Proof. $cx^3+cxxz+cxzz$
$-cxxz-cxzz-cz^3$

$cx^3 \quad 0 \quad 0 \quad -cz^3$

The terms of the dividend should always stand according to the powers of some quantity most regularly repeated, and which is likewise in the divisor, where it must stand in the first place.

$2a+x \;)\; 12a^3+6aax-8axx-4x^3 \;(\; 6aa-4xx$
$12a^3+6aax-8axx-4x^3$

0 0 0 0

When the dividend is not divided exactly by the divisor, then the division may be carried on to any number of terms, in the same manner as in decimals; thus unity, divided by $1-x$, gives $1+x+xx+x^3+$, &c. which may be continued at pleasure, by adding constantly the powers of x to the quotient, whose exponents are the natural numbers.

Operation. $1-x \;)\; 1 \;(\; 1+x+xx+x^3+$ &c.
$1-x$

$0+x$
$x-xx$

$0+xx$
$xx-x^3$

$0+x^3$

If

If the divisor is $1+x$, the quotient will be $1-x-xx-x^3-$, &c.

As division is very intricate in many cases, the reader should multiply compound quantities together, and then divide the product by one of the factors, till he is pretty perfect therein.

Of the SQUARE *and* CUBE ROOT.

The square of $a+b$ is $\overline{a+b} \times \overline{a+b} = aa+2ab+bb$; the cube of $a+b$ is $\overline{a+b} \times \overline{a+b} \times \overline{a+b} = a^3+3aab+3aab+b^3$. In the same manner the square of $a-b$ is $\overline{a-b} \times \overline{a-b} = aa-2ab+bb$; and the cube of $a-b$ is $\overline{a-b} \times \overline{a-b} \times \overline{a-b} = a^3-3aab+3abb-b^3$.

In general, the square of any number is the product of that number multiplied by itself; and the cube of any number is the product of its square, multiplied by that number: But the square root of any number is such as, when it is multiplied by itself, produces the square; and the cube root of any number is such a number, which being multiplied by itself, and the product again by that number, produces the cube.

Thus the square of 4 is 16, and the cube 64; the square of 5 is 25, and the cube 125; and, on the contrary, the square root of 16 is 4; the cube root of 64 is 4 also.

To extract the square root of $aa+2ab+bb$, begin to find the root of the first term aa, which is a, whose square aa being subtracted, leaves $2ab+bb$; to find the second term of the root

divide

divide the remainder by $2a$, twice the quotient, which gives $+b$ for the second term of the root, then b being added to the divisor $2a$, gives $2a+b$, and the sum multiplied by the last term b of the quotient, gives $2ab+bb$; this product subtracted from the remainder $2ab+bb$, leaves nothing; whence we conclude, that the square root of $aa+2ab+bb$, is $a+b$. For $a+b$, multiplied by itself, produces that square.

Operation. $aa+2ab+bb \mid a+b$

aa

—

0

$2a \mid 2ab+bb$

$2ab+bb$

———

$0 \quad 0$

If any remainder is left, the operation must be continued, by dividing always with twice the quotient, as before; thus, if the square root of $aa+2ab+bb+2ac+2bc+cc$ be required; then after having found the two first terms $a+b$, as before, the remainder $2ac+2bc+cc$ must be divided by $2a+2b$ twice the root, which gives c for the third term of the root; c being added to the divisor $2a+2b$, gives $2a+2b+c$, and this sum, multiplied by the last term c, of the root, gives $2ac+2bc+cc$, which being subtracted from the remainder $2ac+2bc+cc$, leaves nothing. Therefore the square root of $aa+2ab+bb+2ac+2bc+cc$, is $a+b+c$.

Operat. $aa+2ab+bb+2ac+2bc+cc \mid a+b+c$

aa

—

0

$2a \mid 2ab+bb$

$2ab+bb$

———

$0 \quad 0$

$2a+$

$2a+2b \;)\; 2ac+2bc+cc$
$\quad\; 2ac+2bc+cc$

o o o

The ſquare root of any number is found after the ſame manner; but as it is neceſſary to know the ſquares or cubes of the numbers under ten, they may be found by this table.

1	2	3	4	5	6	7	8	9	10	Roots.
1	4	9	16	25	36	49	64	81	100	Squares.
1	8	27	64	125	216	343	512	729	1000	Cubes.

Thus the ſquare root of any number 31.36, is found by ſeparating the figures with a point, two by two, from the right to the left; then as the ſquare next to 31, of the two firſt, is 25, whoſe root is 5; ſet it down as the firſt figure of the root, and the ſquare 25, being ſubtracted, leaves 6, to which join the two next figures 36; divide the remainder 636 by 10, twice the quotient 5, which gives 6 for the ſecond figure of the quotient; join 6 to the diviſor 10, and multiply 106 by 6; then the product 636 being ſubtracted from the remainder 636, leaves nothing. Hence 56 is the root ſought: For 56, multiplied by 56, gives the ſquare 3136.

```
31.36 ( 56
 25
-----
10) 6.36
    6.36
-----
 ...
```

The ſquare root of the annexed number is found by conſidering that the neareſt ſquare to 32 is 25, whoſe root 5 being ſet to the quotient, and the ſquare ſubtracted leaves 7; to this remainder join the two next figures 14, and divide the ſum 7.14 by 10, twice the quotient 5, which gives

```
32.14.89 ( 567
 25
-----
10) 7.14
    6 36
-----
112) 78 89
     78.89
-----
 ....
```

6 for the next figure of the quotient; join 6 to the diviſor, then the ſum 106, multiplied by 6, gives 636, this ſum ſubtracted from the remainder 714, leaves 78, to this join the two laſt figures 89, and divide the ſum 78.89 by 112, twice the quotient 56, then you will find 7 for the laſt figure of the root; join 7 to the diviſor 112, and multiply 1127 by 7, and ſubtract the product 7889 from the laſt remainder, which leaves nothing. Therefore 567 is the ſquare root ſought.

If there are more figures in the ſquare whoſe root is to be found, the operation is continued in the ſame manner; that is, every time you take down two figures, you always divide with twice the quotient, and then the figure found is joined to the diviſor and the ſum multiplied by the ſame figure: When the root cannot be found exactly, by adding any number of cyphers to the remainder, and continuing the operation, you will have half as many decimals in the root as cyphers were added.

To extract the cube root of $a^3+3aab+3abb+b^3$ find the root of the firſt term a^3, which is a, ſubtract its cube and there remains $3aab+3abb+b^3$; ſquare the root a, and multiply it by 3, which gives $3aa$ for the diviſor, then $3aab$, divided by $3aa$, gives $+b$ for the ſecond term of the root; multiply the diviſor $3aa$, by the term $+b$, which gives $3aab$, and multiply $3a+b$, the ſum of three times the firſt term of the root added to the ſecond by the ſquare of the laſt b, which gives $3abb+b^2$; this product, added to the product $3aab$, found before, and the ſum $3aab+3abb+b^3$, ſubtracted from the remainder $3aab+3abb+b^3$, leaves nothing: Therefore $a+b$ is the root required. If there be any remainder

mainder, the operation is to be continued, dividing always by three times the ſquare of the root found, and multiplying the diviſor by the quotient; by adding three times the firſt figures of the root to the laſt, and the ſum being multiplied by the ſquare of the laſt, then the ſum of the two products is ſubtracted from the remainder.

In the ſame manner the cube root of any number is found; for if the figures are ſeparated three by three from the right to the left with points, ſuch as in the annexed example; find the cube next to 2, which is 1; ſet 1 as the firſt figure of the root, and ſubtract it from 2; to the remainder 1 join the three next figures 460, and divide

```
                2.460.375 (135 Root.
                1
              ---------
3) 1.460          33
     9             9
    297          -----
  -----          297
   1197
                  395
507) 263.375       25
    2535          -----
     9875         1975
    -----         790
   263375        -----
  --------       9875
  .......
```

the ſum 1.460 by 3, three times the ſquare of the root 1, which gives 3 for the ſecond figure of the root, and the product of the diviſor 3, multiplied by the quotient 3, gives 9; then 3 times 1, joined to 3, gives 33, this multiplied by the ſquare 9, of the laſt figure, gives 297; now adding this product to the former, in ſuch a manner that the laſt figure 7 ſtands under the laſt figure 0 of the remainder, and the ſum being ſubtracted from the remainder, leaves 263, to which join the three laſt figures 375, and divide the ſum 263.375 by 507, three times the ſquare of the root 13; the quotient 5 being multiplied by the diviſor 507, gives 2535 for the firſt ſum; then if to 39, three times the firſt figures 13, of the root,

the laſt 5 is joined; and the ſum 395, multiplied by 25, the ſquare of the laſt, the product will be 9875, which being added to the former 2535, in ſuch a manner that the laſt figure 5 ſtands under the laſt 5 of the remainder, and the ſum ſubtracted from the remainder, leaves nothing; whence 135 is the cube root required.

If there be any more figures, or the root cannot be found exactly, by adding cyphers to the remainder; the operation is continued in the ſame manner as before; that is, you divide always by three times the ſquare of the root; and to the product of the diviſor, multiplied by the laſt figure of the root, you add the product of three times the firſt figures of the root joined to the laſt, multiplied by the ſquare of the laſt.

The proof of this operation is, the riſing the root to the power of which it is the root; but as this is very operoſe in large numbers, the operation may be proved as follows. Conſider the ſingle values of the figures only, without any regard to their places, take off as many nines as you can from the root, and raiſe the remainder, leſs than 9, to the power of the root, from this power take alſo as many nines as you can; then if the remainder, leſs than 9, is equal to the remainder of the power or propoſed number, found in the ſame manner as in the root, the operation is right.

Thus 14 is the cube root of 2744; the cube of the ſum 5 is 125, whoſe remainder is 8, as well as that of the cube 2744: Therefore both remainders being 8, the operation is right.

Of FRACTIONS.

A fraction conſiſts of two parts, the one being placed above the other, with a ſtroke between them,

them, as $\frac{z}{3}$, $\frac{b}{a}$, or $\frac{a+b}{c}$.

The under part shews in how many equal parts the quantity is divided, and is called the *Denominator*; the upper part shews how many of these parts are expressed by the fraction, and is called the *Numerator*.

Thus, in the fraction $\frac{3}{4}$, the denominator 4 shews that the quantity is divided into four equal parts; and the numerator 3, that this fraction expresses three of them.

When the numerator is greater than the denominator, as in $\frac{4}{3}$ or $\frac{7}{5}$, the fraction is then called improper; and hence any whole number may be considered as an improper fraction whose denominator is unity.

To reduce a FRACTION *into its lowest Expression.*

Divide both numerator and denominator by their greatest common divisor, then the quotient will be the fraction required.

Examp. $\frac{6}{12}$, divided by 6, gives $\frac{1}{2}$, $\frac{ab}{ac}$ divided by a, gives $\frac{b}{c}$; $\frac{16}{240}$, divided by 8, gives $\frac{1}{30}$; $\frac{ab+ac}{ad}$, divided by a, gives $\frac{b+c}{d}$; and $\frac{aa+2ax+xx}{a+x}$, divided by $a+x$, gives $a+x$.

This rule is evident; because a quantity divided and multiplied by the same number, neither increases nor diminishes its value; for $\frac{na}{nb}=\frac{a}{b}$ $\frac{12}{20}=\frac{1}{6}$.

Fractions should always be reduced to their lowest expressions before any other operation is performed,

ed, becauſe they become thereby leſs operoſe and perplexed. As a fraction may often be reduced to its loweſt expreſſion, without finding the greateſt common diviſor, we ſhall here give ſome particular inſtances.

All numbers ending with a 5 or 0, are diviſible by 5; all numbers whoſe ſums make any number of 9's, without a remainder, when their ſingle values are only conſidered, are diviſible by 9; and if the ſums of any numbers make any number of 6's, then if they end with even numbers they are diviſible by 6, if uneven by 3.

Thus the fraction $\frac{36}{48}$ is diviſible by 9, becauſe both numerator and denominator make 9's exactly, which, being performed, gives $\frac{6}{72}$; this fraction is diviſible by 9 for the ſame reaſon, by which it is reduced to its loweſt expreſſion $\frac{1}{8}$.

As in the fraction $\frac{11}{12}\frac{12}{90}\frac{18}{06}$ both numerator and denominator make 9's exactly, they are diviſible by 9, which done, gives $\frac{1}{14}\frac{18}{34}\frac{2}{4}$; theſe numbers making 6's exactly, and ending in even numbers, are diviſible by 6, by which the fraction is reduced to its loweſt expreſſion $\frac{2}{23}\frac{1}{5}\frac{7}{0}$.

To find the greateſt common Diviſor *of any two numbers.*

Divide the greateſt by the leaſt; divide the leaſt by the remainder, and the laſt diviſor always by the laſt remainder, till you find a number that divides exactly, which will be the diviſor ſought.

Examp. To find the greateſt common diviſor of 2346 and 89046, divide the laſt by the firſt and there remains 2244, divide the diviſor 2346 by this remainder, which leaves 102; divide again the laſt diviſor 2244 by the laſt remainder 102, which leaves nothing: Therefore 102 is the diviſor

diviſor ſought. For 2349, divided by 102, gives 23, and 89046 gives 873.

The greateſt common diviſor of algebraic expreſſions is found in the ſame manner, only obſerving, *Always to divide the expreſſions by their ſimple diviſors if they have any, and then proceed as before.*

Thus in the expreſſions $ac+cx$, and $aax+2axx+x^3$, divide the firſt by c, and the ſecond by x; and then the quotient $aa+2ax+xx$ by the quotient $a+x$, which gives $a+x$ for the diviſor ſought.

If both expreſſions have a ſimple common diviſor, divide them therewith; and if they have alſo a compound common diviſor, then the product of theſe two common diviſors will be the required one.

Thus $aax-x^3$, and $aax-2axx+x^3$, being divided by x, and the quotient $aa-2ax+xx$ by the quotient $aa-xx$, leaves $-2ax+2xx$; this being divided by $-2x$, and the laſt diviſor $aa-xx$ by the quotient $a-x$, leaves nothing: Therefore the product $x \times \overline{a-x}$, or $ax-xx$, is the diviſor ſought. For $aax-x^3$, divided by $ax-xx$, gives $a+x$, and $aax-2axx+x^3$, gives $a-x$.

To prove this rule let m and n be any two numbers, whoſe common diviſor is required; if m, divided by n, quotes a and leaves p; n divided by p quotes b and leaves q; and p divided by q quotes c, without any remainder; then as the product of the quotient multiplied by the diviſor, added to the remainder, is equal to the dividend, we have $m=na+p$, $n=pb+q$, $p=qc$; and as q divides p exactly by ſuppoſition, it will alſo divide the multiple pb of p, as well as itſelf; and becauſe q divides $pb+q$, it will divide its equal

n, or any multiple na of n; therefore it will divide $na+p$, or its equal m; and consequently q divides both m and n exactly. Now because q is the first common divisor found, and the remainders constantly diminish, it follows that q is the greatest common divisor of m and n.

To find the least Number divisible by any given Number of Figures.

Divide all the numbers that can be, by the same divisor, set down the quotients, divide those quotients and the remaining numbers by some common divisor, and continue so till there remains no numbers that have a common divisor; then the product of all the divisors and the last remaining numbers, will be the required one.

Let a, $3a$, b, $2b$, $3c$, d, express the given numbers.

Then a) 1, 3, b, $2b$, $3c$, d,
b) 1, 3, 1, 2, $3c$, d,
3) 1, 1, 1, 2, c, d,

Hence $2 \times 3 \times abcd$, or $6abcd$, will be the number sought. For 6 is divisible by 2 and 3, and $abcd$ by a, b, c, d: And it is plain that this number is the least that is divisible by the given numbers.

To reduce Fractions *under the same* Denomination.

Multiply the several numerators separately by all the denominators, excepting its own, then the products will be the numerators, and the product of all the denominators the common denominator.

Examp. The fractions $\frac{2}{3}$, $\frac{1}{7}$, are reduced under the same denomination, by multiplying the numerator 2 by the denominator 7, and the numerator

merator 5 by the denominator 3, which gives the numerators 14, 15, and the product 21 of the denominators 3, 7, the common denominator. Hence $\frac{2}{3} = \frac{14}{21}$, and $\frac{5}{7} = \frac{15}{21}$.

The fractions $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$, are respectively equal to $\frac{adf}{bdf}, \frac{bcf}{bdf}, \frac{bde}{bdf}$.

This rule is evident, since the reducing fractions under the same denomination is no more than to multiply and divide the fraction by the same number, which neither increases nor diminishes its value.

The operation may be shortened when some of the divisors have a common measure, by finding the least number divisible by all the denominators; for *if that number be divided by the denominator of each fraction, and the quotient multiplied by the numerator, you will have the numerator, and the first number will be the common denominator.*

Examp. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{8+12+2+5}{16}$; $\frac{b}{a} + \frac{c}{d} + \frac{d}{3a} = \frac{3bd+3ac+dd}{3ad}$.

To add *or* subtract FRACTIONS.

Reduce them under the same denomination, if they are not so; then the sum or difference of the numerators, divided by the common denominator, will be the fraction required.

Examp. $\frac{2}{3} + \frac{3}{4} = \frac{8+9}{12} = \frac{17}{12}$: $4 - \frac{2}{3} = \frac{12-2}{3}$ $= \frac{10}{3}$: $\frac{a}{b} - c = \frac{a-bc}{b}$: $a - \frac{bd}{e} = \frac{ae-bd}{e}$:

$a +$

$a+\frac{ax}{a-x}=\frac{aa}{a-x}$: $a+x+\frac{ax}{a-x}=\frac{aa+ax-xx}{a-x}$

Obſerve that the denominator of a whole quantity is always ſuppoſed to be unity. As like parts of a quantity may be added or ſubtracted in the ſame manner as whole quantities, this rule is evident: For a half added to a half makes unity, and a third part added to a third part makes two thirds.

To multiply *or* divide FRACTIONS.

In multiplication the product of all the numerators, divided by the product of all the denominators, will be the fraction required; and if the diviſor be inverted, that is, if the numerator be placed inſtead of the denominator, and the denominator inſtead of the numerator, the diviſion becomes a multiplication.

Obſerve that the ſign $\div$ ſtands for diviſion; thus $a \div b$ means that a is to be divided by b.

Examp. $\frac{1}{2}\times\frac{3}{8}=\frac{3}{16}$; $20\times\frac{1}{7}=\frac{20}{7}$; $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf}$; $\frac{1}{2}\div\frac{1}{4}=\frac{1}{2}\times\frac{4}{3}=\frac{4}{6}$. $a\div\frac{b}{c}=a\times\frac{c}{b}=\frac{ac}{b}$; $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.

For if we ſuppoſe $\frac{a}{b}=m$, $\frac{c}{d}=n$; then as the product of the quotient, multiplied by the diviſor, gives the dividend, we have $a=bm$, $c=dn$; and becauſe equals, multiplied by equals, produce equals, $ac=bdnm$; but equals divided by equals, give equal quotients, by dividing ac and

$bdmn$

$bdmn$ by bd, we get $\frac{ac}{bd} = mn$: Therefore $\frac{a}{b}$ multiplied by $\frac{c}{d}$, gives $\frac{ac}{bd}$. Diviſion is proved by multiplication; thus, the quotient $\frac{ad}{bc}$ multiplied by the diviſor $\frac{c}{d}$ gives the dividend $\frac{acd}{bcd}$ or $\frac{a}{b}$ when reduced.

If there are any numbers in the numerators, and in the denominators, that have a common diviſor, *divide as many in the one as in the other, and multiply the quotients as before.*

Examp. $\frac{3}{4} \times \frac{4}{3} = \frac{1}{2}$; $\frac{1}{7} \times \frac{4}{3} \times \frac{3}{4} \times \frac{4}{5} = \frac{1}{5}$; $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d}$: For $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}$, gives $\frac{abc}{bcd}$ by what has been proved before, and $\frac{a}{d}$ when reduced to its loweſt expreſſion.

Examp. $a \div \frac{a}{d} = a \times \frac{d}{a} = d$; $\frac{a+x}{c} \times \frac{d}{a+x} = \frac{d}{c}$; $\frac{a+x}{b} \div \frac{c+x}{d} = \frac{a+x}{b} \times \frac{d}{c+x} = \frac{ad+dx}{bc+cx}$.

The ſquare or cube root of any fraction is found by extracting the root of both numerator and denominator ſeparately; thus the ſquare root of $\frac{4}{9}$ is $\frac{2}{3}$; the cube root of $\frac{8}{64}$ is $\frac{2}{4}$; the ſquare root of $\frac{aa}{bb}$ is $\frac{a}{b}$, and the cube root of $\frac{a^3}{c^3}$ is $\frac{a}{c}$.

For

For $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$, and $\frac{a}{c} \times \frac{a}{c} \times \frac{a}{c} = \frac{a^3}{a^3}$.

To reduce a quantity of a lower denomination into a fraction of a higher, *divide the quantity by the number of times that its denomination is contained in the higher.*

Thus three pence are reduced into a fraction of a ſhilling, by dividing them by 12, the number of pence in a ſhilling, as $\frac{3}{12}$; and 7 ſhillings into a fraction of a pound, by dividing them by 20, as $\frac{7}{20}$. Again, a foot is reduced into a fraction of a yard, by dividing it by 3, as $\frac{1}{3}$; an inch by dividing it by 36, as $\frac{1}{36}$.

To reduce fractions into whole numbers of a lower denomination, *multiply the numerator by the number of times that its denomination is contained in the higher, and divide the product by the denominator.*

The fraction $\frac{7}{8}$ of a pound is reduced into ſhillings by multiplying the numerator 7 by 20, and dividing the product 140 by 8, which gives $17\frac{4}{8}$ ſhillings; and if the numerator 4 be multiplied by 12, and the product 48, divided by 8, it gives 6 pence; therefore the fraction $\frac{7}{8}$ of a pound is 17 ſhillings and 6 pence.

Examp. $\pounds\frac{4}{5} \times 20 = 16\,Sh.$ $\pounds\frac{2}{5} \times 20 = 8\,Sh.$ $Yds.\frac{2}{3} \times 3 = 2\,Feet.$ $Feet.\frac{5}{6} \times 12 = 10\,Inches.$

Obſerve, that when the denominator is equal to the number of parts that the lower denomination is contained in the higher, the numerator will be the number required; thus $\frac{3}{12}$ of a ſhilling is 3 pence; $\frac{12}{20}$ of a pound, 17 ſhillings.

To reduce a fraction into decimals, *annex as many*

many cyphers to the numerator as you want decimals, and divide by the denominator.

Thus $\frac{7}{8}$ is reduced into decimals by joining three cyphers to the numerator 7, and dividing 7000 by 8, which gives .875; the fraction $\frac{1}{32}$, by joining 5 cyphers, and dividing 100000 by 32, which gives .03125; and the fraction $\frac{12}{5}$; by joining one cypher to the numerator 12, and dividing 120 by 5, which gives 2.4.

Decimals are added and subtracted like whole numbers; only care must be taken *to place similar places above one another*, as units above units, and tens above tens; they are multiplied and divided as integers, only *there must be as many decimals in the product as are in both factors*, and *in the quotient, as many as there are in the dividend more than in the divisor*; and the division may be carried on to any degree of exactness by adding cyphers to the remainder.

Of PROPORTIONS.

Quantities of the same kind may be compared together two ways, that is, by considering how much the one exceeds the other; or how often the one is contained in the other.

The comparison, by the excess, is called *Arithmetical*, and is done by subtraction; the comparison by the contents *Geometrical*, and is done by division; thus, to find the difference between a and b, we write $a-b$; and to find how often b is contained in a, we write $\frac{a}{b}$.

An arithmetical *Progression* is a series of terms which exceed each other by the same difference, as 2, 4, 6, 8, &c. or $a+x$, $a+2x$, $a+3x$, &c.

It

It is evident, that any term of an arithmetical progreſſion is found by adding the common difference to the preceding term; thus, if a be the firſt term, and x their difference, then will $a+x$ be the ſecond; $a+x+x$, or $a+2x$, the third; $a+3x$ the fourth; and, in general, $a+\overline{z-1},x$ the z term.

In any arithmetical progreſſion, *the ſum of the firſt and laſt term is always equal to the ſum of any two equidiſtant ones.*

For in a, $a+x$, $a+2x$, $a+3x$, $a+4x$, $a+5x$, the ſum $2a+5x$ of the firſt and laſt is equal to the ſum of the ſecond $a+x$, and fifth $a+4x$, as likewiſe to the ſum of the third $a+2x$ and fourth $a+3x$.

If the number of terms be uneven, the ſame thing will be true, by taking twice the middle term.

The ſum of all the terms of an arithmetical progreſſion, *is equal to the firſt and laſt term added together, multiplied by half the number of terms.*

For becauſe the ſum of any two equidiſtant terms, from the firſt and laſt, is always the ſame, it is evident that the whole ſum is equal to half as many times the ſum of the firſt and laſt as there are terms. If, therefore, a repreſents the firſt term, b the laſt, and z the number of terms, then the ſum of all the terms will be expreſſed by $\overline{a+b}\times\frac{z}{2}$.

Thus, if 1, 2, 3, 4, 5, 6, ; then $\overline{1+6}\times 3=21$; or if 3, 5, 7, 9, 11, 13, 15; then $\overline{3+15}\times\frac{7}{2}=63$.

If the progreſſion begins with 0, then $a=0$, and

and $\frac{bz}{2}$, the ſum; thus 0, 2, 4, 6, 8, 10, 12, gives $\frac{12\times 7}{2}=42$.

If the common difference be unity, then $a+\overline{z-1}$ will be the laſt term by what has been ſaid before; therefore $\overline{2a+z-1}\times\frac{z}{2}$, will then expreſs the ſum. Hence, if the firſt term be unity, as well as the common difference, then will $2a+z-1=z+1$, and the ſum will then be $\frac{z\times\overline{z+1}}{2}$; but if the firſt term is 0, then will $2a+z-1=z-1$, and the ſum $\frac{z\times\overline{z-1}}{2}$.

In a ſeries of numbers, 1, 4, 9, 16, 25, 36, *&c.* where the terms are the ſquares of the natural numbers, 1, 2, 3, *&c.* the ſum of any number of terms z, beginning from the firſt, is equal to $\frac{z\times\overline{z+1}\times\overline{2z+1}}{6}$; that is, *to the product of the number of terms, multiplied by that number, plus one, and that product by twice that number, plus one, the whole being divided by* 6.

For $1+4+9=\frac{3\times\overline{3+1}\times\overline{6+1}}{6}$; $1+4+9+16=\frac{4\times\overline{4+1}\times\overline{8+1}}{6}$; $1+4+9+16+25=\frac{5\times\overline{5+1}\times\overline{10+1}}{6}$

This will always hold good, let there be ever ſo many terms: Thus, if there be twenty terms, then $z=20$, and $\frac{z\times\overline{z+1}\times\overline{2z+1}}{6}=\frac{20\times 21\times 41}{6}=$

2870.

2870. If the first term of the progression is o, then we must substitute $z-1$ for z, in order to have the sum $\frac{z \times \overline{z-1} \times \overline{2z-1}}{6}$. For by diminishing z by unity, we diminish the sum by one term.

In a progression, as $m \times n$, $\overline{m+1} \times \overline{n+1}$, $\overline{m+2} \times \overline{n+2}$, continued to z terms; where the terms are the products of those of any two arithmetical progressions.

If we suppose $m+n=a$, by multiplying the factors, and setting the products under each other, we get

$$\begin{array}{l} mn+0+0 \\ mn+1a+1 \\ mn+2a+4 \\ mn+3a+9 \\ mn+4a+16 \\ \&c.\ \&c.\ \&c. \end{array}$$

Then it is evident, that the sum of all the terms, in the first vertical column, is equal to the product of one of the terms multiplied by z, the number of terms; that is, $=mnz$ and the sum of the second $=\frac{az \times \overline{z-1}}{2}$; and that of the third $=\frac{z \times \overline{z-1} \times \overline{2z-1}}{6}$; and therefore the sum of all the terms of this progression is equal to $mnz + \frac{az \times \overline{z-1}}{2} + \frac{z \times \overline{z-1} \times \overline{2z-1}}{6}$; which proves what has been said in pages 124, 125, and 126.

If of four quantities the quotient of the first and second be equal to the third and fourth, then those quantities are said to be in *geometrical proportion*

portion; such are the numbers 2, 4, 6, 12; and the quantities a, ra, b, br; the geometrical proportion is wrote thus, $a:ra::b:rb$; which in words is, a is to ra, as b is to rb.

Hence, if four quantities are proportional, *the product of the extremes is equal to the product of the means*; for $a \times rb = ra \times b$, or $rab = rab$: And, on the contrary, if four quantities are such, that the product of the means is equal to that of the extremes, they are proportional.

To find a fourth proportional d, to three given quantities a, b, c, *divide the product of the second and third by the first, and the quotient will be the term required*; for we have $ad = bc$, by supposition, and dividing these equals by a, we get $d = \frac{bc}{a}$.

Thus, if to the numbers 4, 6, 12, a fourth proportional be required; then 6, multiplied by 12, and the product 72 divided by 4, gives 18 for the number sought.

If the two middle terms of four proportionals are equal, the proportion is said to be continued, as $a:ar::ar:ar^2$; or as the numbers $4:6::6:9$.

If it be required to find a mean proportional between two given numbers, *extract the square root of their product, and you will have the mean required.* Thus, a mean between 4 and 9 is found by extracting the square root of the product 4×9, or 36, which gives 6 for that mean.

If three quantities a, ar, arr, are in a continued proportion; the square of the first is to the square of the second, as the first is to the fourth. For $aa:aarr::a:arr$.

If four quantities, as a, ar, arr, ar^3, are in a continued proportion, the cube of the first is to the cube of the second, as the first is to the fourth, *viz.* $a^3:a^3r^3::a:ar^3$.

Hence, to find the first of the two mean proportionals between any two given numbers, as 2 and 16; say as 2 is to 16, so is the cube 8 of the first 2 to the cube of the number sought, which gives $\frac{8 \times 16}{2} = 64$, whose cube root 4 is the number required; the other mean is found by extracting the square root of the product 64, of the second 4 and fourth 16, which gives 8: For 2 : 4 : : 4 : 8 : : 8 : 16.

Of EQUATIONS *and their* Reduction.

When a quantity is equal to one or more quantities, the equality is called *Equation*, as $a+b=x$, or $ab=cx$: The value of an unknown quantity is said to be known, when that quantity is on one side of the equation, with a positive sign; and all the known quantities on the other, as in $x=4+6$, wherein x is equal to the sum of 4 and 6.

When an unknown quantity, which we always express by x, y, or z, is involved with known ones in an equation, the several operations required to find the value of the unknown quantity, are called *Reductions*.

REDUCTION *by* ADDITION *and* SUBTRACTION.

If a known quantity is at the same side with an unknown one, *take it away, and transpose it to the other side, with a contrary sign*; thus, if $x-4=6$, we get $x=4+6=10$; or if $x-a=b$, then $x=a+b$.

For to take away a quantity from one side and add it to the other, with a contrary sign, is the same thing as to add equals to equals; and therefore the sums must be equal.

If the unknown quantity is negative, it must be

be tranſpoſed to the other ſide; thus, if $6-x=4$, then $6=x+4$, and $6-4$, or $2=x$; or if $a-x=b$, then $a=b+x$, and $a-b=x$.

Reduction *by* Multiplication *and* Division.

If the unknown quantity is divided by a known one, *take it away, and multiply all the other terms thereby:* And if the unknown quantity is multiplied by a known one, *take it away, and divide all the other terms by it:* Thus, if $4x=20$, then $x=\frac{20}{4}=5$; and if $ax=bc+bd$, then $x=\frac{bc+bd}{a}$.

For to take away a diviſor is the ſame thing as to multiply the quantity by it, and to take away a multiplier, is the ſame thing as to divide the quantity thereby; therefore reduction by multiplication and diviſion, is no more than to divide or multiply equals by equals.

Examp. If $\frac{2}{3}x-4=6$, then will $\frac{2}{3}x=10$ by tranſpoſition; $2x=30$, by multiplication; and $x=15$, by diviſion: If $a-\frac{bx}{c}=d$; then $a-d=\frac{bx}{c}$, by tranſpoſition; $ac-cd=bx$ by multiplication; and $\frac{ac-cd}{b}=x$, by diviſion.

When the unknown quantity is repeated ſeveral times, and is on both ſides of the equation, *bring them all on one ſide and proceed as before:* Thus, if $x+\frac{1}{2}x+\frac{1}{4}x=46-\frac{1}{6}x$, then $x+\frac{1}{2}x+\frac{1}{4}x+\frac{1}{6}x=46$, by tranſpoſition; and reducing the fractions under the ſame denomination,

we

we get $\frac{12x+6x+3x+2x}{12}=46$, and $\frac{23}{12}x$ $=46$, by addition; hence $x=\frac{12\times 46}{23}=24$.

Reduction is alſo performed by extracting the roots, or raiſing quantities to powers; thus, if $xx=25$, by extracting the ſquare root we get $x=5$; or if $xx-4x+4=25$, then the ſquare root of the firſt ſide is $x-2$, and that of the other 5; hence $x-2=5$, or $x=7$.

Note, the ſquare root of a quantity as $a+b$ is repreſented by $\sqrt{a+b}$, or $\overline{a+b}^{\frac{1}{2}}$ and the cube root by $\sqrt[3]{a+b}$, or $\overline{a+b}^{\frac{1}{3}}$.

Thus, if $xx=ab+cd$, then will $x=\sqrt{ab+cd}$, or if $\sqrt{ax-bb}=c$; then $ax-bb=cc$, and $ax=bb+cc$ by tranſpoſition; hence $x=\frac{bb+cc}{a}$ by diviſion.

In any equation as $xx+2ax=bc$, where the firſt ſide is not a perfect ſquare, *add the ſquare aa of half the coefficient* $2a$, *of the unknown quantity*; then $xx+2ax+aa=aa+bc$, and the firſt ſide is a perfect ſquare; therefore $x+a=\sqrt{aa+bc}$, or $x=-a+\sqrt{aa+bc}$.

This is called the perfecting a ſquare, and is very uſeful hereafter. When $xx-2ax=bc$, then $x=a+\sqrt{aa+bc}$, and when $xx-2ax=-bc$, then will $x=a+\sqrt{aa-bc}$, or $x=a-\sqrt{aa-bc}$, according as x is greater or leſs than a, which is always known by the queſtion. Note, when bc is negative it muſt be leſs than aa, otherwiſe the roots are impoſſible.

Thoſe who will make a farther enquiry into Algebra, may read Mr. *Mac Laurin*'s Algebra, being the beſt of the kind.

E L E-

THE ELEMENTS OF *MATHEMATICS.*

BOOK I.

Of GEOMETRY.

SECT. I.

Of *LINES* and *ANGLES.*

DEFINITIONS.

1. GEOMETRY is that part of the Mathematics, which teaches to compare ſuch quantities with each other as have extenſion.

2. Extenſion is diſtinguiſhed into length, breadth, and thickneſs, which are called dimenſions. *Plate* I.

A Term, or Bound, is that which is the extreme of any thing.

3. A Line has only length, as AB, CD.

4. The terms or bounds of a line are called Points, and have no dimenſions.

5. A Surface has length and breadth only, as C. The bounds of a ſurface are lines.

6. A Solid has length, breadth, and thickneſs, as D. The bounds or terms of a ſolid are ſurfaces.

7. A Right Line lies evenly between its terms, or is the ſhorteſt diſtance from one point to another, as A B.

And a Curveline, that which lies unevenly between its terms, as C D.

8. The opening A of two lines B C, D C, which meet in a point C, is called an Angle, and is generally expreſſed by three letters as B C D, with that placed at the angle in the middle.

Fig. 1. 9. If a line C D meets another line A B, ſo as the angles C D A, C D B, on both ſides are equal; that line C D is ſaid to be perpendicular to the line A B.

10. And the equal angles are called Right-Angles.

11. But if a line E D falls obliquely on another line A B, the greateſt angle A D E is called Obtuſe-Angle, and the leaſt E D B, Acute angle.

Therefore, there are three different ſorts of angles; the right angle, the obtuſe angle, greater than a right one, and the acute, leſs than a right one.

12. If two lines A E, B D, meet any other line A C, ſo as to make the angles E A B, D B C, at the ſame ſide equal, they are ſaid to be parallel.

AXIOMS.

1. Quantities which are equal to one and the ſame quantity, are equal to each other.

2. Congruous quantities, or quantities which being laid one on the other, agree in every part, are equal.

3. If

3. If equal quantities are added to, or subtracted from, equal ones, the ſums or differences will alſo be equal.

4. If equal quantities are added to, or ſubtracted from unequal ones, the ſums or differences will be unequal; and that which was greateſt will remain the greateſt.

5. The whole is equal to all its parts taken together.

6. If quantities are equal, their doubles, or halves, or any like parts are equal.

POSTULATUMS.

1. Grant that a right line may be drawn from one given point to another.

2. That a given right line may be produced at pleaſure.

3. That from a given point a line may be drawn either parallel to another, or ſo as to make any given angle with any other given line.

N. B. We ſhall not refer to theſe poſtulatums in the following work, becauſe the reader may eaſily ſee where they are uſed; and we ſhall hereafter call a right line only a line.

THEOREM I.

1. *If a line* ED, *meets obliquely another line* AB, *the angles* EDA, EDB, *made by theſe lines taken together, are equal to two right angles.* Fig. 1.

Let CD be perpendicular to AB, then as the greater angle ADE, exceeds the right angle ADC, by the angle CDE, or by as much as the leſſer angle EDB wants of it; it is evident that the ſum of theſe angles is equal to two right angles.

COR.

2. Hence, if ſeveral lines meet any other line

AB in the same point D, the sum of all the angles, on the same side of that line AB, will be equal to
* *ax.* 5. two right angles; since the whole is equal * to all its parts taken together.

THEOREM II.

Fig. 2. 3. *If two lines* AB, CD, *cross each other, the opposite angles a, c, are equal.*

Because DE is oblique to AB, the sum of the
* *art.* 1. angles *b*, *c* is * equal to two right angles; and since BE is oblique to CD, the sum of the angles *a*, *b*, is also equal to two right angles; therefore the sum
* *ax.* 1. of the angles *b*, and *c* is * equal to the sum of the angles *a* and *b*: by taking away the angle *b*, which
* *ax.* 3. is common to both; the angle *a* will be * equal to the angle *c*.

COR.

4. Hence it is manifest, that if ever so many lines cross each other in the same point, the sum of the angles at that point will be equal to four right angles: Since all the angles on one side of a line are equal to two right angles, as well as all those on the other.

THEOREM III.

Fig. 3 5. *If two parallel lines*, AB, CD, *are crossed by any other line* EF, *the alternate angles* BGF, CHE, *or* AGF, DHE, *are equal.*

* *art.* 3. For the opposite angles *a*, *c*, are * equal, as
* *def.* 12. likewise the angles *c* and *b*, at the same side * by supposition. Therefore, since both the angles *a* and *b*, are equal to the same angle *c*, they are
* *ax.* 1. equal * to each other. By the same reason the alternate angles BGF, CHE, are equal, being each the difference between two right angles and two equal ones.

COR.

COR. I.

6. Hence, if two lines are interſected by any other line, and the alternate angles *a*, *b*, are equal; theſe lines are parallel to each other. For ſince the angle *a* is equal to * its oppoſite one *c*, and to the alternate one *b*, by ſuppoſition; the angles *c* and *b*, at the ſame ſide, being equal to the ſame angle *a*, are equal to one another; conſequently the lines AB, CD, * are parallel to each other.

* *art.* 3.

* *def.* 12.

COR. II.

7. It appears alſo, that if a line EF is perpendicular to a line CD, it is likewiſe ſo to all the parallels AB to that line: For ſince the angle *b* is a right angle by ſuppoſition, its equal *a* will be a right angle likewiſe. Conſequently EF is * perpendicular to both lines AB and CD.

* *def.* 9, 10.

SECT. II.

Of *PLANE TRIANGLES.*

DEFINITIONS.

13. A Figure is a ſpace contained between one or more terms.

14. A plane figure is that which lies evenly between its terms, or agrees with a right line placed on it according to any poſition.

15. A plane figure terminated by three lines is called a Triangle.

16. When the three ſides of a triangle are equal, it is called an Equilateral triangle.

17. When two ſides are only equal, an Iſoſceles triangle.

18. When one angle of a triangle is a right angle, a Right angled triangle.

19. Any ſide of a triangle may be called the Baſe, then the angle oppoſite to it, is called the Vertex.

20. The ſide oppoſite to the right angle, is called the Hypothenuſe.

There are therefore three different ſorts of triangles, the Equilateral, the Iſoſceles, and the Right-angled Triangle.

THEOREM I.

Fig. 4. 8. *If any ſide* AC *of a triangle* ABC *be produced, the external angle* BCD *will be equal to the two internal oppoſite ones* A, B.

Let the line CE be parallel to the ſide AB; then the angles A and *a*, at the ſame ſide are * *def.* 12 * equal; and becauſe the line BC meets two parallel * *art.* 5. lines BA, CE, the alternate angles B and *b* * are equal.

Therefore the ſum of the angles A and B is * *ax.* 3. * equal to the ſum of the angles *a* and *b*; that is, to the external angle BCD.

COR. I.

9. Hence, the external angle BCD will be always greater than either of the internal oppoſite ones, A or B, ſince it is equal to both.

COR. II.

10. Therefore, parallel lines can never meet, tho' produced ad infinitum: Since, whilſt the lines AB, CB, meet, the external angle BCD, will be always greater than the internal angle A, which is contrary to the definition of parallel lines.

COR. III.

11. It appears alſo, that if the internal angles A and

and B are equal, the external angle BCD will be double either of these angles A or B.

THEOREM II.

12. *The three angles of any triangle taken together, are equal to two right angles.*

The external angle BCD is equal * to the two internal opposite ones A and B; by adding the angle ACB to both, the sum of the angles BCD, and BCA, will be * equal to the sum of the three angles A, B, C, of the triangle; but because the line BC falls obliquely on the line AD, the sum of the angles BCD, and BCA, is equal to * two right angles. * *art.* 8. * *ax.* 3. * *art.* 14

Therefore the sum of the three angles of the triangle ABC, which has been proved to be equal to the sum of these two angles, is also * equal to two right angles. * *ax.* 1.

COR. I.

13. Hence, if in two triangles ABC, DEF, two angles A, B, in the one, are equal to two angles D, E, in the other, either together, or each to each, the third C, in the one, will be equal to the third F in the other. For since the three angles of a triangle are * equal to two right angles; and the sum of the angles A and B is equal to the sum of D and E; by supposition, the angles C and F are equal to the differences, between equals subtracted from equals: Therefore * they are equal. *Fig.* 5, 6 * *art.* 12. * *ax.* 3.

COR. II.

14. It is also evident, if one angle of a triangle is either equal to or greater than a right angle, either of the other angles will be less than a right

angle: Since their ſum is either equal or leſs than a right angle.

THEOREM III.

Fig. 5, 6. 15. *If two triangles* ABC, DEF, *have one angle* A, D, *equal each to each, and the adjacent ſides* AC, *to* DF, AB, *to* DE, *are alſo equal, theſe triangles are equal in all reſpects; that is, the third ſide in the one will be equal to the third ſide in the other, and the angles oppoſite to the equal ſides are equal.*

Let the triangle DEF be placed on the triangle ABC, ſo as DE agrees with its equal AB; then the angle D will agree with its equal A, and the ſide DF with its equal AC; and ſince the points E and F agree with the points B and C, the ſide EF will likewiſe agree with the ſide BC. Therefore theſe triangles are congruous, and conſequently they are * equal in all reſpects; that is, the ſide EF is equal to the ſideB C, the angle E to the angle B, and the angle F to the angle C.

* *ax.* 2.

THEOREM IV.

Fig. 7. 16. *In an Iſoſceles triangle* ABC, *the angles* A, C, *oppoſite to the equal ſides* BA, BC, *are equal.*

For if BD biſects the angle B, the triangles ADB, CDB, having the ſides BA and BC equal, by ſuppoſition, the ſide BD common to both, and the angles at B included by the equal ſides equal by conſtruction, they * are equal in all reſpects: Conſequently; the angles A and C, oppoſite to the common ſide BD are equal.

* *art.* 15.

COR. I.

Fig. 8. 17. Hence, the three angles in an equilateral triangle ABC are equal. For becauſe the ſides AB

AB, AC, are equal, the oppoſite angles C and B are equal; and ſince AB and BC are equal, the oppoſite angles C and A are alſo equal: Conſequently the three angles A, B, C, of an equilateral triangle are * equal. * art. 16. * ax. 1.

COR. II.

18. It appears alſo, that the line BD, which biſects the angle B, contained by two equal ſides BA, BC, biſects the oppoſite ſide AC at right angles: For ſince the triangles ADB, CDB, are * equal, the angles at D, oppoſite to the equal ſides BA, BC, are equal; they are therefore * right angles, and the ſides AD, DC, oppoſite to the equal angles at B, are alſo equal. Fig. 7. * art. 16. * def. 10.

COR. III.

19. It follows alſo, that if two right angled triangles ADB, CDB, have the hypothenuſes BA, BC, equal, and any of the ſides BD, they are equal in all reſpects. For becauſe BA, BC, are equal by ſuppoſition, the angles A and C are * equal; and the angles at D being right angles, the angles at B are * equal. Conſequently the ſides BA, BC, and BD, including theſe equal angles being equal, theſe triangles are * equal in all reſpects. * art. 16. * art. 13. * art. 15.

COR. IV.

20. It is alſo manifeſt, that if two Iſoſceles triangles have either an angle at the baſe or at the vertex equal, they are equiangular. If it be the angles at the vertex, the remaining two in the one will * be equal to the remaining two in the other; and as theſe are equal in the ſame triangles their halves * will be ſo too: And if it be one of the angles at the baſe which are equal, the others at the * art. 12 * ax. 6.

the baſe are likewiſe equal; and conſequently, the
* *art.* 13. third in the one equal to * the third in the other.

COR. V.

21. Laſtly, each of the equal angles, in an iſoſceles triangle, will always be leſs than a right angle; and conſequently its difference to two right angles greater than a right angle.

THEOREM V.

Fig. 9.

22. *In any triangle* ABC, *the greateſt angle* B *is oppoſite to the greateſt ſide* AC; *and the greateſt ſide is oppoſite to the greateſt angle.*

In AC, the greateſt ſides, let AD be equal to one of the other ſides AB, and let BD be drawn; then as the ſides AD, AB, are equal, the angles
* *art.* 16. ADB, ABD, are * equal; and becauſe the external angle, ADB, to the triangle BDC, is
* *art.* 9. greater than * the internal angle C, and its equal ABD leſs than the angle ABC, the angle ABC, oppoſite to the greater ſide AC, is greater than the angle C oppoſite to a leſſer ſide AB. The contrary of this is likewiſe true; for if the greateſt angle is oppoſite to the greater ſide, the greater ſide is oppoſite to the greater angle.

COR.

23. Hence, the ſum of any two ſides of a triangle is always greater and the difference leſs than the third ſide: For ſince the angle ADB is leſs
* *art.* 21. than a * right angle, the angle CDB is greater, and the angle C is leſs than the angle ADB; therefore the angle CDB is greater than the angle C, or CBD: Conſequently, the difference DC
* *art.* 22. between the ſides AC and AB is leſs * than the other ſide BC; and if we add AD to DC, and its equal AB to BC, the ſum AC will be leſs

* than

* than the sum of AB and BC, or the sum of the two sides AB and BC is greater than the third side AC. * *ax.* 4.

THEOREM VI.

24. *If two triangles,* ABC, DEF, *have two sides equal each to each, that which contains the greatest angle between the equal sides, will have the greatest base; and the greatest base will be opposite to the greatest angle.* *Fig.* 10, 11.

Let the side DE be equal to AB and EF, equal to BC: If the angle ABH be equal to the angle DEF, and BH equal to EF, join AH, CH; then the triangles ABH, DEF, having the sides AB, DE, and BH, EF, equal by supposition, and the angles included by these equal sides equal, they are equal * in all respects. Now since BC and BH are equal by supposition, the angles BCH, BHC, are * equal; and as the angle ACH is greater than BCH, and AHC less than its equal BHC, the angle ACH will be greater than AHC; and therefore AH, or DF, * greater than AC: Consequently the greatest base DF is opposite to the greatest angle E, included by the equal sides. * *art.* 15. * *art.* 16. * *art.* 22.

The contrary is also evident, for if AH is greater than AC, the angle ACH, opposite to the greater side AH, will be greater than AHC; and therefore the line BH falls without the triangle ABC, and so the angle ABH will be greater than ABC.

If the point C falls on AH, then AH will still be greater than its part AC; and if the angle C falls within the triangle ABH, then because the external angle BCH, to the triangle DCH, is * greater than the internal angle CHD, and * less *Fig.* 12. * *art.* 9.

* *art.* 21. * leſs than the angle DCH; the angle ACH, which is greater than DCH, will be greater than
* *art.* 22. the angle CHA; therefore AH is * greater than AC.

THEOREM VII.

Fig. 5, 6. 25. *If two triangles have one ſide equal each to each, and any two angles in the one are equal to any two in the other, each to each, they are equal in all reſpects.*

If the ſide AB be equal to the ſide DE, the angle A to the angle D, and B to E, then BC will be equal to EF, and AC to DF: For if the triangle DEF be placed on the triangle ABC, ſo that the ſide DE agrees with its equal AB, then becauſe the angle D is equal to the angle A, the ſide DF will fall on the ſide AC; and as the angle E is equal to the angle B, the ſide EF will alſo fall on the ſide BC; and therefore the angle F will agree with the angle C. Conſequently the
* *ax.* 2. * ſide DF is equal to the ſide AC; the ſide EF equal to the ſide BC; and the angle F equal to the angle C.

If the ſide AB be equal to the ſide DE, the angle A to D, and the angles C and F, oppoſite to the given ſides; then becauſe the third angle B in
* *art.* 13. the one is equal to * the third E in the other, theſe triangles are ſtill equal in all reſpects. Conſequently, two triangles which have one ſide equal each to each, and any two angles in the one, equal to any two angles in the other, are equal in all reſpects..

COR. I.

Fig. 7. 26. Hence, if two angles, A, C, of a triangle ABC, are equal, the oppoſite ſides BC, AC, to theſe equal angles, are alſo equal. For if BD be pre-

perpendicular to AC, the right angled triangles ADB, CDB, having beſides the right angles at D, the angles A and C equal by ſuppoſition, the angles ABD, CBD, will be * equal; and ſince * *art.* 13. the ſide BD is common to both, they are * equal * *art.* 25. in all reſpects. Conſequently the ſides AB, BC, oppoſite to the right angles, are equal.

COR. II.

27. It is manifeſt, that if the three angles of a triangle, ABC, are equal, the ſides will be equal likewiſe. For if the angles A and C are equal, the oppoſite ſides BC, BA, are equal; and if the angles A and B are equal, the oppoſite ſides BC, AC, are alſo equal: Therefore the ſide BC being equal to both the ſides BA and AC, they are all three * equal to each other. *Fig.* 8, * *ax.* 1.

THEOREM VIII.

28. *If the three ſides of a triangle* ABC *are equal to the three ſides of another triangle* DEF, *theſe triangles will be equiangular.* *Fig.* 5, 6.

If AB is equal to DE, AC to DF, and BC to EF, then the angle A will be equal to the angle D, the angle B to E, and C to F. For if the angle B is either greater or leſs than the angle E, then becauſe the adjacent ſides to theſe angles are equal by ſuppoſition, the baſe AC, oppoſite to the angle B, will alſo be * greater or leſs than the baſe * *art.* 24. DF, which is contrary to ſuppoſition; therefore ſince the angle B can neither be greater nor leſs than the angle E, it muſt be equal to it. Conſequently * theſe triangles are equal in all reſpects. * *art.* 15. Whence, triangles whoſe ſides are equal reſpectively, are equiangular.

THE-

THEOREM IX.

Fig. 13. 29. *If within a triangle* A B C, *there be described another triangle* A D C *on the same base* A C, *the sum of the sides of the inscribed triangle will be less than the sum of the sides of the other, and the angle which they contain greater than the angle in the other.*

1. Let A D produced, meet B C in E; then be-
* *art.* 23. cause the sum of the sides A B and B E is * greater than A E, by adding C E, the sum of A B and
* *ax* 4. B C will be * greater than the sum of A E and E C;
* *art.* 23. and the sum of the sides C E and E D is * greater than C D, by adding A D; the sum of C E and
* *ax.* 4. E A will be greater than * the sum of C D and D A: Therefore, since the sum of A B and B C is greater than the sum of A E and E C, and this greater than the sum of A D and D C, the sum of A B and B C will be greater than the sum of A D and D C.

2. The external angle A D C to the triangle
* *art.* 9. C D E is greater than * the internal opposite one C E D; and the external angle C E D to the tri-
* *art.* 9. angle A B E, is greater than * the internal one B: Therefore, since the angle A D C is greater than the angle C E D, and this greater than B, the angle A D C will be greater than the angle B.

THEOREM X.

Fig. 14, 15. 30. *If two triangles* A B C, D E F, *have two sides respectively equal, as well as one angle, each to each, these two triangles will be equal in all respects.*

If the side A B is equal to the side D E, B C to E F, the angle A equal to the angle D, these triangles are equal in all respects.

Let

Let BG, EH, be perpendicular to the sides AC and DF; then the right angled triangles AGB, DHE, having, besies the right angles, the angles A and D equal by supposition, the third ABG in the one will be * equal to the third DEH in the other; and as AB, DE, are equal by supposition, these triangles are * equal in all respects; therefore BG is equal to EH, and AG equal DH. * art. 13. * art. 25.

And since the right angled triangles BGC, EHF, have the sides BG, EH, equal, as well as the hypothenuses BC, EF, they are * equal in all respects. Therefore the side GC equal to HF, and the angle C to the angle F; as likewise the * side AC equal to DF, and the angle ABC * equal to the angle DEF. * art. 19. * ax. 3. * art. 13.

If the given angles A, D are acute, and the opposite sides BC, EF, less than the sides AB, EF; the angles C, and F, must be known to be either both greater or both less than a right angle, otherwise this theorem is indetermined.

SCHOLLIUM.

It appears from the articles 15, 25, 28, and 30, that if in two triangles, the three sides, or two sides and an angle, or else one side and two angles, are respectively equal; these triangles are always equal in all respects, only when two sides in one triangle are equal to two sides in the other, and one angle opposite to the equal sides; and these angles are acute and the opposite sides less than the others: it must farther be known, that the angles opposite to the other equal sides, are either both greater or less than a right angle.

SECT.

SECT. III.

Of *PARALLELOGRAMS* and *TRIANGLES.*

DEFINITIONS.

1. A Plane figure of four ſides is called in general, a Quadrilateral.

2. Parallelogram is a quadrilateral, whoſe oppoſite ſides are parallel.

3. Rectangle is a parallelogram, whoſe four ſides are equal.

4. Square, is a parallelogram, whoſe four ſides are equal, as well as the four angles.

5. All other quadrilaterals are called Trapeziums.

6. A Diagonal is the line which joins the oppoſite angles of a quadrilateral.

7. If two lines parallel to the ſides of a parallelogram interſect each other in the diagonal, they form two leſſer parallelograms on each ſide of the diagonal, which are called Complements.

8. The Altitude of a parallelogram is the perpendicular diſtance between the two oppoſite ſides.

9. Either of the ſides to which the altitude is perpendicular is called the Baſe.

10. The Altitude of a triangle is the perpendicular drawn from the vertex to the baſe, and the parts of the baſe terminated by the perpendicular, are called Segments.

Therefore two triangles whoſe baſes are in the ſame right line, which have the ſame vertex, have

alſo

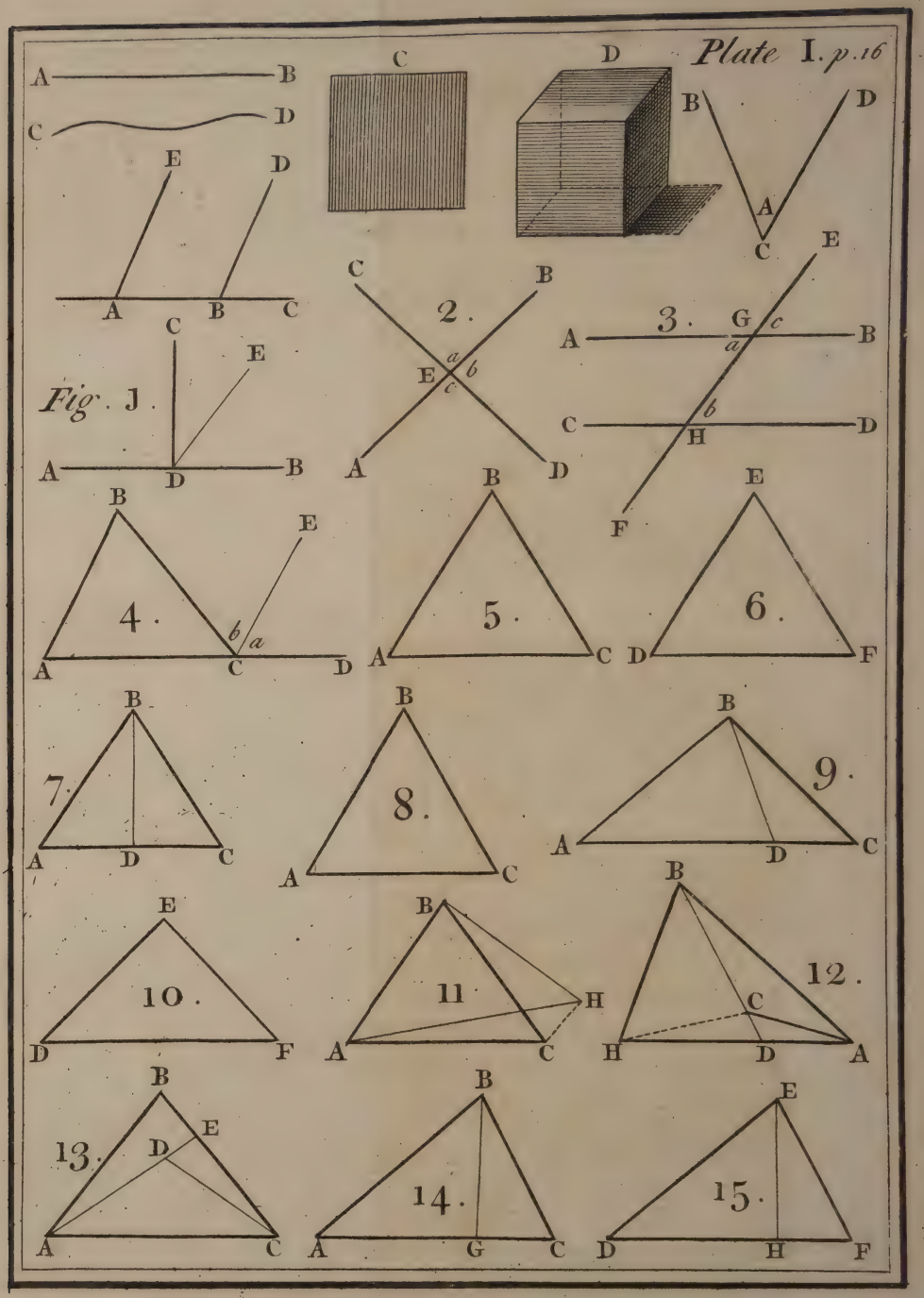
Plate I. p. 16
A B
C D
E D
A B C
Fig. 1.
C E
A D B
C B
2.
a b
E c
A D
C B D
A
B D
A C
3.
G c
a
b
C H D
E
F
4.
B E
b a
A C D
5.
B
A C
6.
E
D F
7.
B
A D C
8.
B
A C
9.
B
A D C
10.
E
D F
11.
B H
A C
12.
B
C
H D A
13.
B E
D
A C
14.
B
A G C
15.
E
D H F

alſo the ſame altitude; for there can be drawn from the ſame point but one perpendicular to the ſame line.

THEOREM I.

31. *The oppoſite ſides of a parallelogram are equal, as alſo the oppoſite angles; and the diagonal biſects the parallelogram.* *Plate* II. *Fig.* 1,

Becauſe the oppoſite ſides AD, BC, and AB, DC, are parallel by definition; the alternate angles ADB, DBC, are * equal, as likewiſe the alternate ones ABD, and BDC. Therefore the triangles ABD, BCD, having the ſide BD common to both, and the adjacent angles equal, are * equal in all reſpects. * *art.* 5. * *art.* 25.

Hence, the ſides AD, BC, oppoſite to the equal angles DBA, BDC, and the ſides AB and DC oppoſite to the equal angles ADB, DBC, are equal; the angles A and C oppoſite to the common ſide DB, are alſo equal, as well as the angles ABC, ADC, which are the ſums of equal ones.

COR. I.

32. As the three angles of any triangle are * equal to two right angles, and the four angles of a quadrilateral equal to the ſum of the angles of two triangles; the four angles of a quadrilateral are * equal to four right angles. * *art.* 12. * *ax.* 1.

COR. II.

33. Since the four angles of a quadrilateral are equal to four right angles, and all the angles of a ſquare or rectangle are equal by definition, it follows, that the angles of a ſquare or rectangle are right ones.

COR. III.

34. Because, the opposite sides of a rectangle are equal and perpendicular to the other opposite ones, which are parallel; it follows, that parallelograms, which are between the same parallels, have the same altitude.

COR. IV.

35. It is manifest, that any triangle ADB is half the parallelogram AC of the same base AD and the same altitude; since, if DC, is parallel to AB, and BC to AD, the triangle ABD is
* *art.* 31. half * the parallelogram AC.

COR. V.

36. Consequently, triangles which are between the same parallels have equal altitudes, since they are the same with those of the parallelograms of the same base and altitude.

THEOREM II.

Fig. 2. 37. *The diagonals* AC, DB, *of a parallelogram bisect each other.*

Let E be their intersection, then as the sides
* *ax.* 31. BC, AD, are * parallel and equal, as also the
* *art.* 5. alternate angles * ECB, EAD, and CBE, EDA; the triangles AED, BEC, having one side equal each to each, with the adjacent angles,
* *art.* 25. are * equal in all respects. Consequently, the sides AE, EC, opposite to the equal angles EDA, EBC, are equal, as well as the sides BE, ED, opposite to the equal angles ECB, EAD.

THEOREM III.

Fig. 3. 38. *The Complements* AI, IC, *of a parallelogram* AC *are equal.*

Because the diagonal BD, divides the parallelo-
* *art.* 31. grams AC, FG, EH, each into two equal * tri-

angles

angles; by taking BFI, IED, from BAD, and their equals BGI, IHD, from BCD equal to BAD; the remainder AI will be equal to the * remainder CI. Consequently the complements of parallelograms are equal. * *ax.* 3.

COR.

39. It is evident, that the triangles BGI, IHD, are equiangular; and as FI is equal to BG, and IE equal to DH; the parallelogram AI, of the sides BG, HD, is equal to the parallelogram CI, of the sides GI and IH, opposite to the same angles, taken alternately.

THEOREM IV.

40. *Parallellograms, which have the same or equal Bases, and are between the same parallels, or, which is the same, have equal altitudes, are equal.* Fig. 4.

First, If BF be parallel to AL, the parallelograms AF, DB, which have the same base AD are equal. For BC, EF, being both equal * to AD, are * equal to each other; by adding CE to both, the sum BE will be * equal to the sum CF; and since AB, DC are parallel and * equal, the angles ABC, DCF, on the same side * are equal: Therefore the triangles ABE, DCF, having the angles at B, C, equal, as well as the adjacent sides, * are equal in all respects. From which taking the same space CGE, and adding AGD, to the remainders ABCG, DGEF, we shall have the parallelogram DB equal to * the parallelogram AF. * *art.* 31. * *ax.* 1. * *ax.* 3. * *art.* 31. * *def.* 12. * *art.* 15. * *ax.* 3.

If the point E falls between the points B and C, there need no more than to add the same space AECD, to the equal triangles ABE, DCF, in

order to have the parallelogram DB equal to the parallelogram AF.

Fig. 5. *Secondly*, If BG be parallel to AH, the parallelograms DB, HF, which have equal bases AD and EH are equal.

For adding DE to the equals AD, EH, we * *ax.* 3. shall have AE equal * to DH; and drawing AF, DG, the triangles AEF, DHG, having the an- * *def.* gles at E and H formed by parallel lines * EF, * *def.* 12. HG, * equal, as well as the adjacent sides, are * *art.* 15. equal * in all respects: Therefore the angles EAF, HDG, are equal as well as the sides AF, DG; and since FG is equal and parallel to AD by supposition, AG is a parallelogram, equal to both parallelograms DB, HF, by case the first; * *ax.* 1. and so the parallelograms DB and HF are * equal. Consequently, parallelograms, which have the same, or equal, bases, and equal altitudes, are equal.

COR. I.

Fig. 4. 41. It is evident, that if DB is a rectangle, any parallelogram AF, is equal to a rectangle DB, which has the same or equal base and the same altitude.

COR. II.

* *art.* 35. 42. Since triangles are the halves * of parallelograms of the same base and altitude, and as all parallelograms of the same or equal bases, and equal altitudes, are equal; it is manifest, that all triangles which have the same or equal bases, and equal altitudes, are also equal; because the whole's * *ax.* 6. being equal, their halves * will be so too.

THEOREM V.

Fig. 6. 43. *In any two right angled triangles, the rectangle, made by the hypothenuse of one and any side*

ſide of the other, will be equal to the rectangle made by the hypothenuſe in the other, and the ſide oppoſite to the ſame angle in the firſt.

Let the triangle ABC be right angled at B, in one of the ſides BC, take CD equal to the hypothenuſe of the other triangle, and draw DE perpendicular to AC; then the triangles ABC, CED, having the angle C common to both, beſides a right angle, are * equiangular; and the triangle CED will be equal to the equiangular triangle whoſe hypothenuſe is equal to the hypothenuſe CD. * *art.* 13.

Now if AF be drawn parallel to CD, and DF, FG, parallel to CA, AB, the latter meeting CB produced in G, then the parallelograms AG, FC, having the ſame baſe AF, and being between the ſame parallel, are * equal; and the parallelogram FC is alſo equal to * the rectangle FE of the ſame baſe FD or AC and the ſame altitude DE. Therefore, ſince both the rectangles AG and FE are equal to the ſame parallelogram FC, they are * equal to one another. Conſequently, in any two right angled equiangular triangles, the rectangle AG, made by the hypothenuſe DC or AF, of the one, and any ſide AB of the other, is equal to the rectangle made by the hypothenuſe AC of the other and the ſide DE in the firſt, oppoſite to the ſame angle C. * *art.* 40. * *art.* 41. * *ax.* 1.

COR.

44. Hence, if CD is equal to AB, the figure AG will be the ſquare of AB; and if from the right angle B, a line BL be drawn perpendicular to AC, then the right angled triangles CED, ALB, having the angles C and A common with the triangle ABC, are both * equiangular to it, and ſo are equiangular to each * other; and becauſe * *art.* 13. * *ax.* 1.

their hypothenuſes CD, AB, are equal by ſuppo-
* *art.* 25. ſition, they are * equal to each other; and therefore DE and AL oppoſite to the equal angles C, ABL, are equal. Conſequently, if in a right angled triangle ABC there be drawn a perpendicular to the hypothenuſe AC, the ſquare AG of any ſide AB, will be equal to the rectangle EF, made of the hypothenuſe AC, and the ſegment AL (equal to DE) of the baſe next to that ſide.

THEOREM VI.

Fig. 7. 45. *If a line* AD *be divided into any two parts* AE,ED, *the ſquare* ABCD, *of the whole line, will be equal to the ſum of the ſquares and to two rectangles of the parts.*

Draw the diagonal AC; the line EF parallel to DC, and thro' its interſection I with the diagonal, GH parallel to BC; then becauſe AD and DC are equal by ſuppoſition, the angles
* *art.* 16. DAC, DCA, are * equal; and ſince EI is pa-
* *def.* 12. rallel to DC, the angle EIA is equal * to the angle EAI; and hence the ſides AE, EI, are
* *art.* 26. * equal as well as HI and HC. Therefore EG is the ſquare of AE, and HF that of ED; and DI, IB, the rectangles of AE or EI and ED. Conſequently the ſquare DB, of the whole line, is
* *ax.* 5. * equal to the ſum of the ſquares EG, HF, and to the two rectangles DI, IB, of the parts AE, ED.

THEOREM VII.

Fig. 8, 9. 46. *The difference between any two ſquares* ABCD, ILCK, *is equal to the rectangle made by the ſum and difference of their ſides.*

In AD produced, take DG equal to the ſide LC or ED of the ſquare KL, produce IK, and draw GF perpendicular to AG; then becauſe DG

DG is equal to DE by conſtruction, AG will be equal to the ſum of the ſides; and as HF is parallel to AG, AH is equal to the difference; whence the rectangle AGFH is made by the ſum and difference of the ſides; and as DG is equal to DE by conſtruction, the rectangle DF is equal to the rectangle DI, or equal to its * complement IB. Conſequently, the difference HD and HL between the ſquares ABCD and ILCK is equal to the rectangle AF, made by the ſum and difference of their ſides.

* *art* 38.

COR.

47. It is alſo manifeſt, that if a line EG be equally divided in D, and unequally in A, the rectangle AHFG, made of the unequal parts AE, AG, will be equal to the difference between the ſquares DB, KL, made of half that line ED and the part AD. For ſince ED or LC is equal to DG by ſuppoſition, as is AE, AH, by conſtruction, the rectangle DF is equal to the rectangle KE or HL. Conſequently, the rectangle AHFG is equal to the difference DH and HL between the ſquares DB and KL.

This corollary is very uſeful in Conic-Sections, for which reaſon it ought to be remembered.

THEOREM VIII.

48. *The rectangle* AC *of any two lines* AD *and* AB, *is equal to the ſum of all the rectangles made by one of theſe lines* AB, *and the parts of the other* AD, *divided any how.* Fig. 10.

Let the line AD be divided into any parts AE, EG, GD, through theſe points of diviſion draw EF, GH, parallel to AB; then it's evident, that the rectangles AF, EH, GC, which are made by the line AB and the parts of AD,

have all the same altitude, and therefore are equal
* *ax.* 5. to * AC. Consequently, the rectangle made by any two lines is equal to the sum of all the rectangles made by one of these lines, and the parts of the other divided any how.

THEOREM IX.

Fig. 11. 49. *In any right angled triangle* ACB, *the square* ABED, *of the hypothenuse* AB, *is equal to the sum of the squares* CH, CF, *of the other two sides.*

If through the right angle C, the line LG be drawn parallel to AD, then because the square
* *art.* 44. CH of AC is equal to the rectangle LD of the hypothenuse AB or AD, and the segment AL; and the square CF, of BC, equal to the rectangle LE, of the hypothenuse BA or BE, and the segment BL: Therefore, since these two rectangles LD, LE, are equal to the square ABED, of the hypothenuse AB; it follows that in any right angled triangle, the square of the hypothenuse is equal to the sum of the squares of the two other sides.

THEOREM X.

Fig. 12, 13. 50. *In any triangle* ABC, *where a perpendicular* BD *is drawn to the base* AC, *the rectangle made by the sum and difference of the sides* AB, BC, *is equal to the rectangle made by the sum and difference of the segments* AD, DC, *of the base.*

The perpendicular BD divides the triangle into two right angled triangles ADB, CDB; and
* *art.* 49. the square of AB is equal to * the sum of the squares of AD and DB; and the square of BC is equal to the sum of the squares of CD and BD, subtracting the latter equality from the former; then because the square of BD,

sub-

ſubtracted from itſelf, leaves nothing, the difference between the ſquares of AB and BC will be equal to the difference between the ſquares of AD and DC. But the differences of ſquares * are equal to rectangles made by the ſum and difference of their ſides. Conſequently, in any triangle the rectangle made by the ſum and difference of the ſides AB, BC, is equal to the rectangle made by the ſum and difference of the ſegments AD, DC. *art. 46.

THEOREM XI.

51. *Any Trapezium, which has two ſides* AD, BC, *parallel, is equal to a rectangle of the ſame height, and whoſe baſe is equal to half the ſum of the two parallel ſides.* Fig. 14.

If EF be parallel to AB, and ſo as to biſect DC in G, and meets BC produced in F, then becauſe the alternate angles D, C, and E, F, are * equal, as well as the ſides DG, GC, the triangles DEG, GFC, are * equal in all reſpects. Therefore the parallelogram ABFE is equal to the Trapezium ABCD; but AE or BF is equal to half the ſum of the parallel ſides AD, BC; and this parallelogram is equal to a rectangle of the ſame * baſe and altitude. Conſequently, the trapezium is * equal to the rectangle of the ſame altitude, and whoſe baſe is equal to half the ſum of the parallel ſides. * *art.* 5. * *art.* 25. * *art.* 41. * *ax.* 1.

SECT.

SECT IV.

Of PROPORTIONS.

This ſection is divided into two parts, the firſt contains the definitions and general theorems, which comprehend chiefly what is ſaid in the fifth book of *Euclid*; the ſecond, the application of proportions to lines and ſurfaces, ſuch as in his ſixth book.

PART THE FIRST.

DEFINITIONS.

1. A Part of a magnitude is a magnitude, which meaſures the whole exactly.

2. Therefore Parts of a magnitude will be a magnitude, which is meaſured by the ſame part as the whole.

For inſtance, 3 A *is a part of* 12 A; *and* 6 A, 9 A, 12 A, 15 A, *are parts of* 18 A; *ſince* 18 A *and theſe parts are meaſured by* 3 A.

3. Equal Parts of two or more magnitudes, are the ſame number of equal parts taken of them all.

Thus equal parts of 20 A *and* 30 B, *are* 6 A, 9 B, *or* 16 A, 24 B; *becauſe* 6, 9, *are the three tenths of* 20 *and* 30, *and* 16, 24, *the four fifths: Or, in general, equal parts of two magnitudes* A, B, *may be repreſented by* n A, *and* n B, *when* n *expreſſes any common fraction: And it may be ob-*

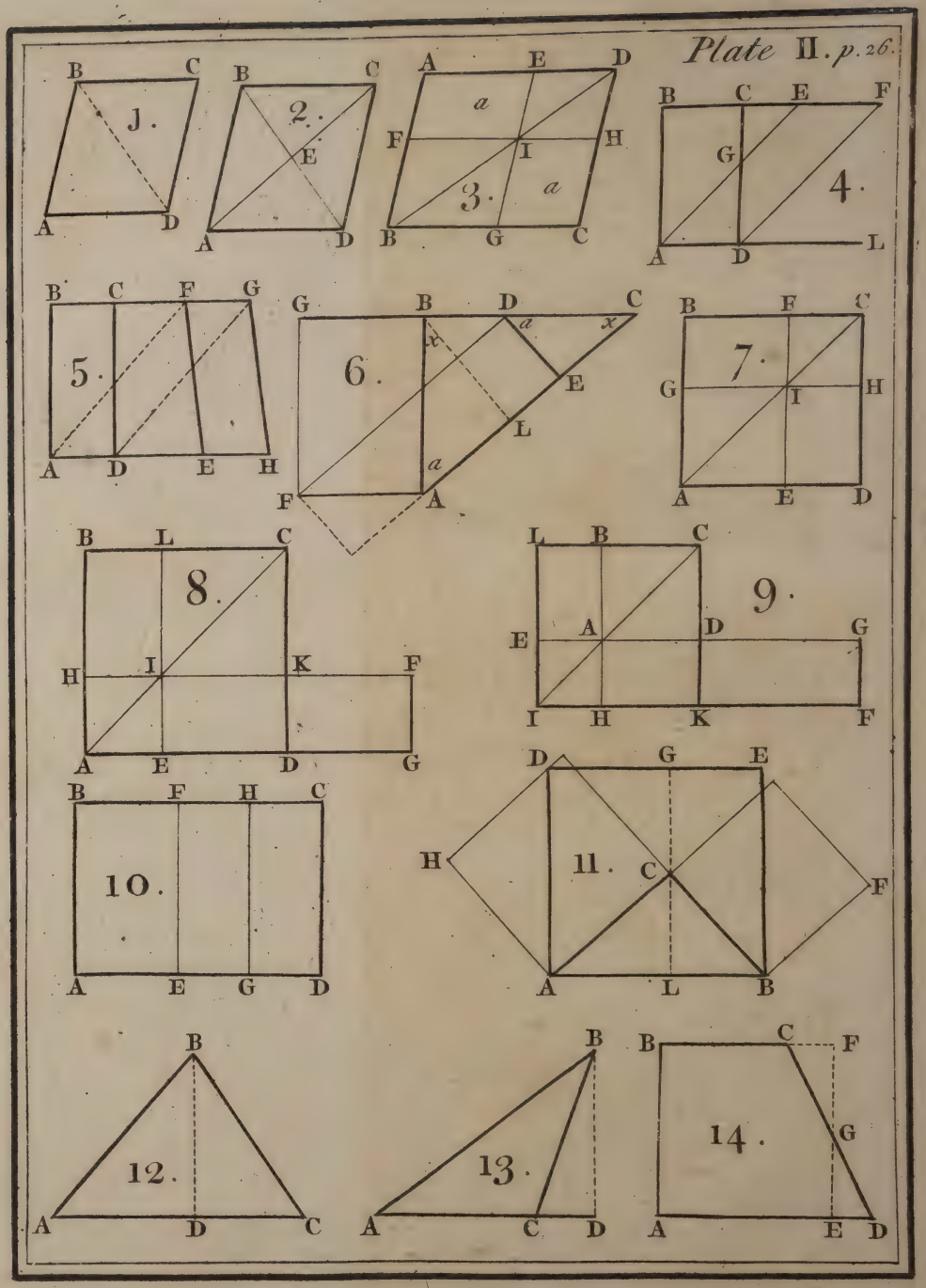
Plate II. p. 26.
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obſerved, that parts of a magnitude may be leſs, equal to, or greater than the magnitude itſelf.

4. Ratio, is the relation between two magnitudes of the ſame kind, with reſpect to quantity.

In the comparing two magnitudes A and B, the firſt A is called Antecedent, and the ſecond B Conſequent.

In commenſurable magnitudes the ratio is found by dividing the antecedent by the conſequent. As for example, the ratio of 20 A *and* 5 A, *is* $\frac{20}{5}$, *or* 4; *and in general the ratio of* nA *and* mA *is* $\frac{n}{m}$.

5. Four magnitudes A, B, C, D, are ſaid to be in the ſame ratio or proportion, the firſt A to the ſecond B, and the third C to the fourth D; when any equal parts whatſoever nA, nC of the antecedents compared with their conſequents B, D, reſpectively, are either both together, leſs, equal to, or greater than their reſpective conſequents.

6. Magnitudes which have the ſame ratio are called Proportionals, and are marked thus, A:B::C:D, which ſignifies that A is to B, as C is to D.

7. Magnitudes are ſaid to be in a continued proportion, when the conſequent of any ratio is always equal to the antecedent of the ſucceeding ratio: As here A:B::B:C::C:D: the continued proportion is alſo marked thus ∴ A:B:C:D.

8. If magnitudes are in a continued proportion as ∴ A:B:C::D, the ratio of the firſt A to the third C is ſaid to be Duplicate, and the ratio of the firſt to the fourth D Triplicate, to that of the firſt A and ſecond B.

On

On the contrary, the ratio of the firſt A to the ſecond B, is ſaid to be ſubduplicate to the ratio of the firſt A and third C; and ſubtriplicate to that of the firſt A and fourth D.

9. If there be two proportions each of four magnitudes, ſuch as A : B :: C : D, and B : E :: F : G, wherein the firſt conſequent B in the firſt is equal to the firſt antecedent B in the ſecond; then the ratio of the firſt antecedent A of the firſt, to the firſt conſequent E of the ſecond, is ſaid to be compound of the ratios of C to D, and of F to G, of the other four terms.

This definition ſeems to be more agreeable to the words of *Euclid*'s 23d *prop. book* VI. where he uſes compound ratios, than thoſe generally given by the tranſlators.

10. Equal magnitudes are in the ratio of equality, inequal magnitudes in the ratio of inequality; the greater to the leſs in the ratio of the greater inequality, and the leſs to the greater in the ratio of the leſſer inequality.

This definition requires an explanation; firſt it muſt be obſerved that equal ratios and the ratio of equality are two diſtinct things; for when four magnitudes are proportional, it is ſaid that their ratios are equal; but when equal magnitudes are compared together, they are ſaid to be in the ratio of equality: Whence, the ratio of equality is properly the unit of the ratios of all magnitudes of the ſame kind, and equal ratios are either greater or leſs than that of equality. In the next place we obſerved, after the definition of ratios, that in commenſurable magnitudes the ratio is found by dividing the antecedent by the conſequent: If therefore the antecedent is greater than the conſequent, the ratio of the antecedent to the conſequent muſt

must be greater than that of the consequent to the antecedent. Thus, if 20 A is the antecedent, and 10 A the consequent, the ratio of 20 A to 10 A is 2, and that of 10 to 20 is $\frac{1}{2}$.

11. Inverse Ratio, is when the consequent is taken as antecedent, and the antecedent as consequent; thus if A : B : : C : D, then B : A : : D : C, is inversely.

12. Alternate Ratio, is when the antecedent of one ratio is compared to the antecedent of another, and the consequent to the consequent; as if A : B : : C : D, then A : C : : B : D, is alternately. But this comparison can only be made when the four magnitudes are of the same kind.

Notes of Abbreviation.

A = B, signifies, that A is equal to B.
A + B, the sum of A and B.
A — B, their difference.
A B, a rectangle whose sides are denoted by A and B.
A A, the square whose side is denoted by A.

N. B. *Though letters are used to express magnitudes the reader is not to imagine, that we intend to demonstrate the doctrine of proportion by algebraic computations: As it is necessary to fix upon some symbol or other to represent magnitudes, in order to assist the mind, in distinguishing one from another, letters appear to be more easy than lines, as* Euclid *expresses them. But the reader may, if he pleases, mark lines over each letter, if he thinks it to be more convenient.*

SCHOL-

SCHOLLIUM.

It is proper to obſerve, that there are three ſigns in the 5th definition of proportion, by which proportionality is diſcovered, becauſe the compariſon of magnitudes may be diſtinguiſhed into three caſes; for they may be always commenſurable or incommenſurable, or may be ſometimes commenſurable and at others incommenſurable, as in plain figures.

When magnitudes are commenſurable, if any equal parts of the antecedents are equal to their reſpective conſequents, that ſign only is ſufficient to prove the proportionality: Since, if any other equal parts of the antecedents are taken, they muſt neceſſarily be both together greater or leſs than the conſequents.

And when magnitudes are incommenſurable, as the radius to the circumference, it is ſufficient to prove, that any equal parts of the antecedents, when compared with the reſpective conſequents, are both together always greater or leſs than the conſequents. For were it poſſible, that any parts whatſoever of one antecedent could be equal to its conſequent, the ſame parts of the other antecedent would likewiſe be equal to its conſequent; ſince it is manifeſt, that they can neither be greater nor leſs, unleſs the others are alſo greater or leſs.

But when magnitudes may either be commenſurable or incommenſurable, it is plain, that the three ſigns muſt be proved.

Hence it is evident, that our definition comprehends all poſſible caſes that can happen in the compariſons of all kinds of magnitudes whatſoever.

A X-

AXIOMS.

1. If equal parts of magnitudes are equal, the magnitudes themſelves are equal; that is, if $nA = nB$, then $A = B$.

2. If equal parts of magnitudes are unequal, the magnitudes are unequal; and that whoſe parts are greater will be greater: Thus, if nA is greater than nB, A is greater than B.

3. Unequal magnitudes, compared with the ſame magnitude, have unequal ratios.

THEOREM I.

52. *Two magnitudes* A, B, *which have the ſame ratio to a third* C, *are equal. If* A : C :: B : C, *I ſay* A = B.

If poſſible let A be greater than B; then becauſe unequal magnitudes cannot have the ſame ratio, when compared with the ſame magnitude, the ratio of A to C is not equal to the ratio of B to C, which is contrary to ſuppoſition; ſince therefore A can neither be greater nor leſs than B, they muſt conſequently be equal.

THEOREM II.

53. *Two ratios which are equal to a third, are equal to each other. If* A : B :: E : F, *and* C : D :: E : F; *I ſay* A : B :: C D.

By taking any equal parts nA, nE, nC, of the antecedents; then if $nA = B$, nE will be $= F$, and $nC = D$, by defin. 5; if nA is greater than B, nE is greater than F, and nC is alſo greater than D; and if nA is leſs than B, nE is leſs than F, and nC leſs than D. Therefore, ſince the equal parts nA, nC, of the antecedents, in A : B :: C : D, when compared with the conſequents, are both together, leſs, equal to, or greater than their reſpective

ſpective conſequents, theſe magnitudes are proportional by defin. 5.

THEOREM III.

54. *Two magnitudes* A, B, *are in the ſame ratio as any of their equal parts. I ſay* rA : rB :: A : B.

By taking any equal parts nr A, nA, of the antecedents; then if nrA = rB, it is evident * that nA = B; and if rnA, is either greater or leſs than rB: nA will alſo * be greater or leſs than B. Conſequently theſe magnitudes are proportional, by defin. 5.

* *ax.* 1.
* *ax* 2.

COR.

55. Hence, if A : B :: C : D; then becauſe nA : nB :: A : B; and mC : mD :: C : D, we have nA : nB :: mC : mD, by * equality.

* *art.* 53.

THEOREM IV.

56. *If four magnitudes,* A, B, C, D, *are proportional; I ſay, if the antecedents* A, C, *are equal, the conſequents* B, D, *are equal; or if the firſt antecedent* A *is either greater or leſs than the ſecond, its conſequent* B *will always be greater or leſs than the conſequent* D.

Becauſe A : B :: C : D; if A = C, then will * B alſo be = D; and if A be greater than B, it is evident that B can neither be equal nor leſs than D; ſince if B = D, then A muſt be * equal to C; but A is ſuppoſed greater than C, therefore B cannot be equal to D, nor leſs; it muſt conſequently be greater: By the ſame argument, if A is leſs than C, B is alſo leſs than D.

* *art.* 52.
* *art.* 52.

THEOREM V.

57. *If four magnitudes,* A, B, C, D, *are proportional, they will alternately be ſo. If* A : B :: C : D; *I ſay* A : C :: B : D.

By

By taking any equal parts nA, nB, of the antecedents; then becauſe * nA : nB : : C : D; and it has been * proved, that when the antecedents nA, C, are equal, the conſequents nB, D, are equal; or if the firſt antecedent nA is either greater or leſs than the ſecond C, its conſequent nB is alſo greater or leſs than the conſequent D. Since therefore the four quantities A, C, B, D, are ſuch, that any equal parts nA, nB, of the firſt and third, when compared with the ſecond and fourth, are leſs, equal to, or greater than the ſecond and fourth reſpectively, they are proportional by definition 5.

* *art.* 55.
* *art.* 56.

COR.

58. Hence if A : B : : C : D, then alternately A : C : : B : D, and nA : nC : : A : C; or nA : nC : : B : D, by * equality; or alternately nA : B : : nC : D. By the ſame argument, A : nB : : C : nD, and by equality nA : mB : : nC : mD.

* *art.* 53.

THEOREM VI.

59. *If four magnitudes* A, B, C, D, *are proportional, they will inverſely be ſo. If* A : B : : C : D. *I ſay* B : A : : D : C.

Becauſe * nA : mB : : nC : mD; if we conſider nA, nC, as any equal parts of the antecedents, when nA = mB, then * nC = mD; and when nA is greater or leſs than mB, nC is alſo greater than mD; and therefore when mB is greater, equal to, or leſs than nA, mD is likewiſe greater, equal to, or leſs than nC.

* *art.* 58.
* *def.* 5.

Now if we conſider mB, mD, as any equal parts of the antecedents in B : A : : D : C, or in B : nA : : D : nC; then, becauſe it has been proved, that when mB is greater, equal to, or leſs than nA, mD was likewiſe greater, equal to, or leſs than

* *def.* 5. nC; it is evident that * theſe magnitudes are proportional. Conſequently, if four magnitudes are proportional, they will inverſely be ſo.

THEOREM VII.

60. *If four magnitudes* A, B, C, D, *are proportional, they will be ſo by addition. If* A : B :: C : D; *I ſay,* A : A + B :: C : C + D.

By taking any equal parts nA, nC, of the ante-
* *art.* 58. cedents, then will * nA : B :: nC : D : Now, it
* *art.* 56 has been * proved, that if the antecedents are equal, the conſequents are equal; or if the antecedents are unequal, the conſequents are unequal:
* *ax.* 1. Therefore if nA = nC, *or A = C, then B = D; and ſo A + B = C + D; if nA is greater than nC, or
* *ax.* 2. * A greater than C, B is greater than D, and the ſum of A + B greater than the ſum C + D of the
* *ax.* 2. leſſer; and if nA is leſs than nC, or * A leſs than C, B is leſs than D, and the ſum of A + B of the leſſer is leſs than the ſum C + D of the greater. Since therefore the four magnitudes A, A + B, C, C + D, are ſuch, that any equal parts nA, nC, of the firſt and third, when compared with the ſecond and fourth, are greater, equal to, or leſs than the third and fourth reſpectively, they are proportional. Conſequently, if four quantities are proportional, they are ſo by addition.

THEOREM VIII.

61. *If any number of magnitudes,* A, B, C, D, E, F, *are proportional, any antecedent is to its conſequent as the ſum of all the antecedents is to the ſum of all the conſequents. If* A : B :: C : D :: E : F. *I ſay,* A : B :: A + C + E : B + D + F.

Becauſe C : D :: E : F, we have alternately
* *ax.* 60. C : E :: D : F, and * C : C + E :: D : D + F; or
* *art.* 57. alternately * C : D :: C + E : D + F; but A : B :: C : D,

by

by ſuppoſition: Therefore $A:B::C+E:D+F$, by * equality; or alternately $A:C+E::B:D+F$, and by addition $A:A+C+E::B:B+D+F$; or alternately * $A:B::A+C+E:B+D+F$. * art. 53. * art. 57.

THEOREM IX.

62. *If four magnitudes*, A, B, C, D, *are proportional, they will be ſo by ſubtraction. If* $A:B::C:D$; *I ſay* $A:B-A::C:D-C$.

For if this proportion $A:B-A::C:D-C$ be true, it will be ſo by * addition: But $A:B-A+A::C:D-C+C$, or $A:B::C:D$. Conſequently, if four magnitudes are proportional, they will be ſo by ſubtaction. * art. 60.

COR.

63. Since $A:C::B-A::D-C$, alternately, and * $A:C::A+B:C+D$ alternately, we have $A+B:C+D::B-A:D-C$, by * equality, or $A+B:B-A::C+D:D-C$. * art. 60. * art. 53.

It is to be obſerved, that B—A, D—C, are the differences between the antecedents and their reſpective conſequents; and therefore, when the antecedents are greater than their conſequents, theſe differences are A—B, C—D; for ſince magnitudes which are proportional are inverſely ſo, the demonſtration in *art.* 61, is the ſame in both caſes.

THEOREM X.

64. *The ſums of proportionals are proportional. If* $A:B::C:D$, *and* $E:F::G:H$, *I ſay*, $A+E:B+F::C+G:D+H$.

By taking any equal parts nA, nC, nE, nG, of the antecedents, it is evident, that when $nA=B$ and $nE=F$, then $nC=D$ and $nG=H$, by ſuppoſition; there will alſo be $nA+nE=B+F$, and $nC+nG=D+H$;

and when these parts are greater or less than their respective consequents, the sums will also be greater or less than the sums of their consequents: Therefore, the sums of proportionals are * proportional. The same thing holds true in respect to their differences.

DEFINITIONS.

1. Right-lined figures are said to be Similar, when all the angles in the one are equal to all the angles in the other, and the sides between the equal angles are proportional, respectively.

2. The sides between the equal angles are said to be Homologous.

PART THE SECOND.

THEOREM XI.

Plate III. *Fig.* 1. 65. *Parallelograms* AE, BF, *or triangles which have the same or equal altitudes, are in the same ratio as their bases. I say*, AB:BC::ADEB:BEFC.

Take BK equal to any part $\frac{3}{2}$AB of AB, draw KL parallel to CF, then because parallelograms which have equal bases and equal altitudes * *art.* 40. are * equal, the parallelogram BL will be the same part $\frac{3}{2}$AE of AE, as the base BK is of the base AB. But it is evident, that if BK is = BC, the parallelogram BL is equal to the parallelogram BF; and if BK is less than BC, BL is also less than BF; or if BK is greater than BC, BL is likewise greater than BF. Consequently, parallelograms which have the same or equal altitudes, are * *def.* 5. in the * same ratio as their bases.

And because the triangles AEB, BEC, are half the parallelograms BD, BF, and the halves * *art.* 35. * are in the same proportion as their whole's: Triangles

angles which have the same or equal altitudes, are in the same ratio of their bases.

COR.

66. It is also manifest, that triangles BEC, BIC, or parallelograms which have the same or equal bases, BC, are to each other as their altitudes BE, BI.

THEOREM XII.

67. *If in a triangle* ABC, *there be drawn a line* DE *parallel to one of the sides* BC, *it will divide the others proportionally. I say*, AD:DB::AE:EC. Fig. 2.

Draw the lines CD, BE; then because the triangles ADE, DEB, have the same vertex E, or the same altitude, they are as their * bases; that is, ADE:DEB::AD:DB; and the triangles ADE, EDC, having the same vertex D, or altitude, are also as * their bases, that is, ADE:EDC::AE:EC. But the triangles DEB, DEC, having the same base DE, and are between the same parallels DE, BC, are * equal. Consequently, AD:DB::AE:EC, by equality * of ratios. * art. 65. * art. 6. * art. 42. * art. 53.

COR.

68. It is manifest, that if the sides of a triangle are divided proportionally, the line joining the points of division will be parallel to the other side. For since * AD:DB::ADE:DEB, and likewise AE:EC::ADE:DEC; but AD:DB::AE:EC, by supposition; therefore ADE:DEC::ADE:DEB, by equality of ratios; Whence * DEC=DEB. And as these equal triangles have the same base DE, they must also have * the same altitude. Consequently, * DE is parallel to the side BC. * art. 65. * art. 53. * art. 42. * art. 36.

THEOREM XIII.

69. *Equiangular triangles* ABC, GHL, *are similar*; *I say*, AB:GH::AC:GL::BC:HL.

In AB take AD=GH, and draw DE parallel to BC; then becauſe the angles A and G are equal, as alſo the angles B and H, by ſuppoſition; and the angle D is equal to the angle B by the de-
* *ax.* 1. finition of parallels; the angle D is equal * to the angle H. Therefore the triangles ADE, GHL, having the ſides AD, GH, equal by conſtruction, and the adjacent angles to theſe equal ſides
* *art.* 25. equal, are * equal in all reſpects: Therefore
* *art.* 67. * AD:DB::AE:EC; or by addition, * AB:
* *art.* 60. AD::AC:AE, and AB:GH::AC:GL; becauſe AD=GH, and AE=GL.

If BD had been taken equal to HG inſtead of AD, and the line DE drawn parallel to AC, it might have been proved in the ſame manner as before, that AB:GH::BC:HL. Conſequently, equiangular triangles are ſimilar.

THEOREM XIV.

70. *If two triangles* ABC, GHL, *have one angle* A, G *equal each to each, and the ſides adjacent to theſe equal angles proportional, they are equiangular and ſimilar.*

In AB, take AD=GH, and in AC, AE=GL, join DE; then the triangles ADE, GHL, having the angles A, G, equal by ſuppoſition, and the adjacent ſides to theſe equal angles
* *art.* 15. equal by conſtruction, they are * equal in all reſpects; and ſince GH:AB::GL:AC, or AD:AB::AE:AC, by ſuppoſition; the triangle ADE, or its equal GHL, is ſimilar to
* *art.* 68. * the triangle ABC.

THE-

THEOREM XV.

71. *If two triangles* ABC, GHL, *have their respective sides proportional, they are similar: Suppose* GH : AB :: GL : AC :: HL : BC.

In AB take AD=GH, and in AC, AE=GL, join DE; then because GH : AB :: GL : AC, by supposition, and AD : AB :: AE : AC, by construction, the line DE is * parallel to BC: And since GH : AB :: HL : BC, by supposition, and * AD : AB :: DE : BC, then because GH = AD by construction, we have HL : BC :: DE : BC, by equality of ratios HL is * equal to DE. Consequently, the triangles GHL, ADE, having all their sides equal, are * equal in all respects, and similar * to the triangle ABC.

* *art.* 68.
* *art.* 67.
* *art.* 52.
* *art.* 28.
* *art.* 68.

THEOREM XVI.

72. *If four lines* A, B, C, D, *are proportional, the rectangle of the first* A *and last* D *will be equal to the rectangle made of the means* B, C. Fig. 3.

Take AB=A, draw BC perpendicular to AB and equal to B; finish the rectangle BD; in AB take AE=C; draw EG parallel to BC; and through its intersection I, with the diagonal draw HF, parallel to AB. Then because * AB : BC :: AE : EI, and A : B :: C : D; the three terms in the first proportion being equal to the three first in the second, the fourth EI will be equal to the fourth D. Now because the complements BI, ID, of parallelograms are * equal, by adding EH to both; the rectangle ED, of the means B, C, is equal to the rectangle BH, of the extremes A, D.

* *art.* 67.
* *art.* 38.

COR. I.

73. Hence, if three lines, A, B, D, are proportional, the square of the mean B is equal to the rectangle of the extremes A, D; for if in the last

conſtruction AE(C) is equal to BC(B), ED will be the ſquare of the mean B, and equal to the rectangle BH, of the extremes A and D.

COR. II.

74. It is alſo manifeſt, that if four lines A, B, C, D, are ſuch, that the rectange BH, of the firſt A and laſt D, is equal to the rectangle ED of the means B, C; theſe lines are proportional, for we have AB : BC :: AE : EI.

COR. III.

75. Hence, all that has been proved of magnitudes in general, in the firſt part of this ſection, follows here in regard to lines.

Directly, - - A : B :: C : D.
Inverſely, - - B : A :: D : C.
Alternately, - - A : C :: B : D.
By multiplicat. $\begin{cases} nA : nB :: C : D. \\ nA : B :: nC : D. \end{cases}$
By addition, - - A : A + B :: C : C + D.
By ſubtraction, - A : A — B :: C : C — D.
By equality, - - A + B : A — B :: C + D : C — D.

* *art.* 72. For ſince * AD = BC, the rectangles of the means will be equal to the rectangle of the extremes. Conſequently, theſe lines are * proportional. * *art.* 74.

THEOREM XVII.

Fig. 4. 76. *If three lines are in a continued proportion, the ſquare of the firſt is to the ſquare of the ſecond as the firſt line is to the third.*

Let in the triangle ABC right angled at C; the perpendicular CL to AB be produced, ſo as LG be = AB; compleat the rectangle AG, as likewiſe

wise the square AF. Then because the right angled triangles ACB, ALC, having the angle A common to both, are * equiangular and similar; * therefore AL : AC :: AC : AB: But the rectangle AG is * equal to the square of AC the mean; and the rectangle AG is to the square AEFL, as * AE to AD, that is, as AL to AB, by construction. Consequently, if three lines AL, AC, AB, are in a continual proportion, the square AF of the first will be to the square AG of the second AC, as the first line AL is to the third AB.

* *art.* 13.
* *art.* 69.
* *art.* 44. or 73.
* *art.* 65.

COR.

77. Hence, squares are in a duplicate proportion to that of their sides, since the ratio of the first A to the third C is duplicate to that * of the first A to the second B by defin. 8.

* *def.* 8.

THEOREM XVIII.

78. *If a line* BD *bisects any angle* B *of a triangle* ABC, *it will divide the opposite side* AC *into parts proportionally to the adjacent sides*; *I say*, AD : DC :: AB : BC.

Fig. 5.

In AB produced, take BE = BC, join EC, then because BE, BC, are * equal, the angles BEC, BCE, are each half the * external angle ABC, and therefore equal to the angle ABD by supposition: Hence, CE is parallel to DB, by definition of parallel lines. Therefore * AD : DC :: AB : BE or BC.

* *art.* 16.
* *art.* 11.
* *art.* 67.

THEOREM XIX.

79. *If the bases* AB, BC, *of two similar parallelograms* BD, BF, *which have the same altitude, are proportional to the bases* GK, LO, *of two other similar parallelograms* GI, LN, *these parallelograms are proportional*; *I say*, BD : BF :: GI : LN.

Fig. 6, 7.

* For

* *art.* 65. For * BD : BF :: AB : BC, and GI : LN :: GK : LO ; and since AB : BC :: GK : LO, by supposition, it follows that BD : BF :: GI : LN,
* *art.* 53. by equality of * ratios.

THEOREM XX.

80. *Similar parallelograms, made of proportional lines, are proportional, if* A : B :: C : D, *and* E : F :: G : H ; *I say,* AE : BF : CG : DH.

* *art.* 65. For * AE : BF :: CE : DF, and by the same reason CE : DF :: CG : DH : Therefore AE :
* *art.* 53. BF :: CG : DH, by equality of * ratios.

COR.

81. Hence, *First*, if four lines A, B, C, D, are proportional, their squares will be proportional ; for if in the last we suppose E, F, G, H, to be equal to A, B, C, D, respectively, we shall have AA : BB :: CC : DD.

Secondly, If six lines A, B, C, D, E, F, are proportional, the square of any antecedent A, is to the square of its consequent B, as the rectangle of the other antecedents is to the rectangle of their consequents : For if A : B :: C : D, and A : B :: E : F,
* *art.* 82. then will * AA : BB :: CE : DF.

THEOREM XXI.

82. *If four rectangles* AE, BF, CG, DH, *or squares, or two squares and two rectangles, are proportional, their sides will also be proportional* ; *I say,* A : B :: C : D, *and* E : F :: G : H.

For suppose A : B :: x : D, and E : F :: G : y,
* *art.* 80. then will * AE : BF :: Gx : Dy : But AE : BF :: CG : DH, by supposition ; therefore CG : Gx :: DH : Dy, by equality of ratios ; or
* *art.* 65. * C : x :: H : y ; and if C = x, then * will H = y. Consequently, A : B :: C : D, and E : F :: G : H.

THE-

THEOREM XXII.

83. *If there be two ranks* A, B, C, *and* D, E, F, *of three continued proportionals, and the firſt* A *is to the laſt* C, *in one, as the firſt* D *is to the laſt* F, *in the other: I ſay, that the firſt* A *is alſo to the ſecond* B *in the one, as the firſt* D *is to the ſecond* E *in the other. If* A : C :: D : F; *I ſay,* A : B :: D : E.

For * A A : B B :: A : C; and by the ſame reaſon D D : E E :: D : F. Now if A : C :: D : F, then will A A : B B :: D D : E F, by * equality of ratios, or * A : B :: D : E. * art. 76. * art. 53. * art. 82.

COR.

84. Hence, if the duplicate ratio of two lines A, B, is equal to the duplicate ratio of two other lines D, E, theſe lines are proportional themſelves, *viz.* A : B :: D : E.

THEOREM XXIII.

85. *If there be two ranks* A, B, C, D, *and* E, B, C, F, *of four proportionals, and either the two means in the one are equal to the two means in the other; or the extremes in the one are equal to the extremes in the other, the other lines will be reciprocally proportional; if* A : B :: C : D, *and* E : B :: C : F. *I ſay,* A : E :: F : D.

For becauſe A : B :: C : D, and E : B : C : F, by ſuppoſition, we have * have A D = B C, and E F = B C. Therefore * A D = E F, by * equality; and * A : E :: F : D. * art. 72. * art. 52. * art. 74.

COR.

86. Hence, if in two ranks, A, B, C, and D, B, E, of three continued proportional lines, the mean in the one is equal to the mean in the other, the extremes will be reciprocally proportional, viz. A : D :: E : C.

THE-

THEOREM XXIV.

Fig. 8, 9. 87. *In equiangular triangles* ABC, DEF, *if perpendiculars are drawn from the equal angles* B, F, *to the oppoſite ſides, they will be proportional to the ſides upon which they fall, viz.* BG : EH :: AC : DF.

For the right angled triangles AGB, DHE, having the angles A, D, equal by ſuppoſition,
* *art.* 13. are * equiangular; whence * AB : DE :: BG : EH;
* *art.* 69. but AB : DE : AC : DF, by ſuppoſition; therefore BG : EH :: AC : DF, by equality of ratios.

THEOREM XXV.

88. *Similar triangles* ABC, DEF, *are to each other as the ſquares of their homologous ſides* AC, DF.

On AC, DF, make the ſquares AK, DM, and draw the diagonals CI, FL; then the trian-
* *art.* 65. gles ABC, ACI, having the ſame baſe AC,* are as their altitudes, that is, ABC : ACI :: BG : AI or AC; and the triangles DEF, DFL, having the ſame baſe DF, are alſo as their altitudes, viz. DEF : DFL :: EH : DL or DF.
* *art.* 87. But * BG : AC :: EH : DF; therefore ABC :
* *art.* 53. ACI :: DEF : DFL, by equality * of ratios; or
* *art.* 54. alternately ABC : DEF :: (ACI : DFL. :) * AK : DM.

COR. I.

89. Hence, becauſe the ſquares AK, DM,
* *art.* 77. are in a * duplicate ratio to that of their ſides; and ſince the ſimilar triangles, ABC, DEF, are in the ſame proportion as theſe ſquares; it is evident, that ſimilar triangles are in a duplicate ratio, to that of their homologous ſides.

COR. II.

Fig. 12. 90. Since parallelograms, AC, EG, are double

ble * the similar triangles ABD, EFH, of the same base and altitudes, it follows that similar parallelograms are in a duplicate ratio to that of their homologous sides AD, EH. * art. 35.

THEOREM XXVI.

91. *Triangles* ABC, DEF, *which have an angle* A, D, *equal each to each, are to each other as the rectangles of the adjacent sides to these equal angles.* Fig. 10, 11.

Let AI=AB, and DL=DE, the rest of the construction being the same as before; then because * ABC:ACI::BG:AI or AB, and DEF:DFL::EH:DL or DE. Now the right angled triangles AGB, DHE, having the angles A and D equal by supposition, they are * equiangular; whence * BG:AB::EH:DE. Therefore ABC:ACI::DEF:DFL, by equality of ratios, or alternately ABC:DEF (::ACI:DFL)::AK:DM. * art. 65. * art. 13. * a.t. 69.

COR. I.

92. Since the parallelograms AC, EG, are double the triangles ABD, EFH, of the same base and altitudes; and since the wholes are in the same proportion * as their halves, it is evident, that two parallelograms AC, EG, which are equiangular, are to each other as the rectangles of their sides. Fig. 12. * art. 54.

COR. II.

93. Hence, parallelograms or triangles, having an angle A, E, equal each to each, and the adjacent sides to these equal angles, reciprocally proportional; viz. AD:EH::EF:AB, are equal. For they are to each other as the rectangles * made by AD, AB, and EH, EF; and as these rectangles are * equal, the parallelograms or triangles will likewise be equal. * art. 92. * art. 72.

THE-

THEOREM XXVII.

Fig. 13. 94. *All similar rectilinear figures, are to each other as the squares of their homologous sides.*

If the angles of the figure A B C D E, are equal to all the angles of the figure K F G H I, each to each, in the order they stand, and the sides between these equal angles, proportional; they will be to each other as the squares of the homologous sides A E, K I.

By drawing lines from the equal angles C, G, to the equal ones A, K, and E, I; then because A B : B C :: K F : F G, and the angles B, F, equal by supposition; the triangles A B C, K F G, * *art.* 70. are * similar; and so the angles B A C, F K G, will be equal as well as their differences C A E, G K I, to the equal angles B A E, F K I.

Now because of the similar triangles A B C, K F G, we have A B : K F :: A C : K G, and A B : K F :: A E : K I, by supposition; therefore A C : K G :: A E : K I, by equality of ratios. And since the angles C A E, G K I, included between the proportional sides, have been proved to be equal, the triangles C A E, G K I, are similar.

Again, since D C : D E :: H G : H I, and the angles D, H, are equal by supposition, the triangles D E C, H I G, are likewise similar.

Therefore, since similar triangles are to each * *art.* 81. other as the squares * of their homologous sides,

$$\text{We have} \begin{cases} \text{ABC} : \text{KFG} :: \overline{\text{AC}}^2 : \overline{\text{GK}}^2 \text{ or as } \overline{\text{AE}}^2 : \overline{\text{KI}}^2 : \\ \text{ACE} : \text{KGI} :: \overline{\text{AE}}^2 : \overline{\text{KI}}^2 . \\ \text{CDE} : \text{GHI} :: \overline{\text{CE}}^2 : \overline{\text{GI}}^2 : \text{ or as } \overline{\text{AE}}^2 : \overline{\text{KI}}^2 . \end{cases}$$

Con-

Consequently, ABCDE : KFGHI :: $\overline{AE}^2$: $\overline{KI}^2$: Since the whole are in the same proportion * as their parts. * art. 54.

Whatever the number of the sides may be, the figures may be always divided into as many similar triangles, wanting two, and the whole's being always proportional to their parts. All similar rectilinear figures are to each other as the squares of their homologuous sides.

COR.

95. Hence, since squares are in the duplicate proportion * to that of their sides, and all similar rectilinear figures being as the squares of their homologuous sides; all similar rectilinear figures are in the duplicate proportion to that of their homologous sides. * art. 77.

SECT. V.

Of the *CIRCLE.*

DEFINITIONS.

1. A Circle is a plane figure bound by one continued curveline called Circumference, being every where equally distant from a point O within called the Center. *Plate* IV. *Fig.* 2

2. A line OA, drawn from the center to the circumference is called a Radius.

3. Any line as AB, terminated by the Circumference is called a Chord.

4. A chord divides the circle into two parts AEB, ACDB, called segments.

5. The chord CD which passes through the center, is called Diameter. 6. Any

6. Any part A E B, of the circumference, is called an Arc.

Any part A O B E, of the circle, terminated by two radii and an arc, is called a Sector.

7. A line which only touches a circle in one point, and being produced both ways, falls without the circle, is called a Tangent.

8. The Measure of an angle is the arc, described from the angular point as center, with any radius, and terminated by the lines which form the angle.

COR.

96. It follows from the definition of the circle, that all the *radii* of the same circle are are equal, as likewise all the diameters, * which are each double the radius. * ax. 6.

POSTULATUM.

Grant that from a given point in a plane, with a given radius, or opening of the compass, the circumference of a circle may be described.

THEOREM. I.

97. *In the same circle, equal chords* A B, C D, *subtend equal arcs*; *and the chords of equal arcs are equal.* Fig. 2.

First, Let the segment C E D be placed on the segment A F B, so as the chord C D agrees with its equal A B, then the arc C E D will agree with the arc A F B; for if the equal chords B F and D E are drawn, as well as the *radii* O B, O E, O D, O F, to their extremities, the triangles O B F, O D E, having all their sides equal, are equal in * all respects. Therefore the point E will agree with the point F; and as this will happen with regard to any points equally * art. 28.

distant

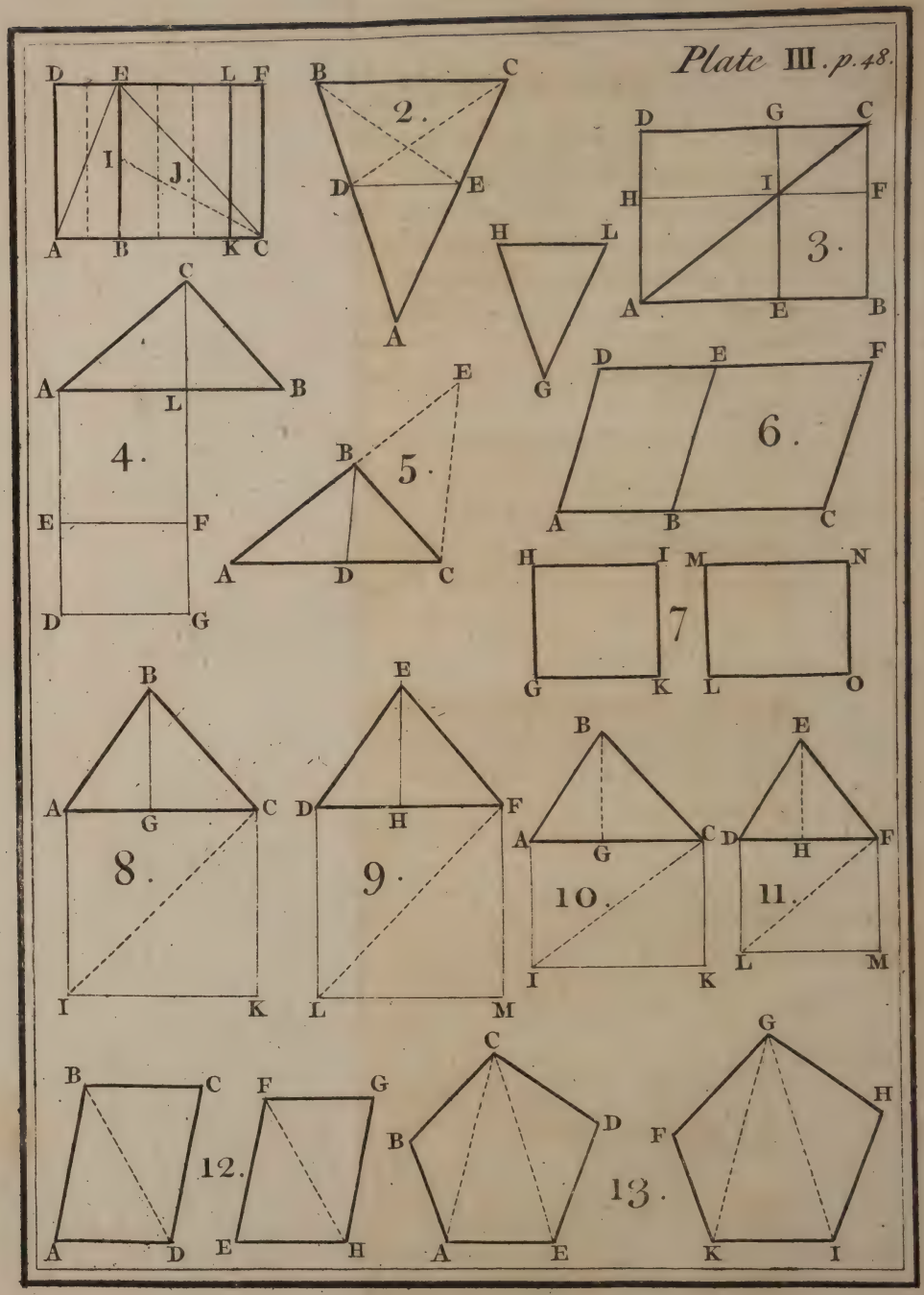
Plate III. p. 48.
2.
3.
4.
5.
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7
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10.
11.
12.
13.

distant from the points B and D, it follows, that the arc C E D agrees every where with the arc A F B. Consequently, equal chords subtend * equal arcs. * *ax.* 2. 1.

Secondly, Since every part of the arc C E D, agrees with its equal A F B, when the point C agrees with the point A, the point D will agree with the point B; and therefore the chord C D will agree with A B. Consequently, the chords of equal arcs are * equal. * *ax.* 2. 1.

COR. I.

98. Hence, the radii drawn to the extremities of equal arcs, in the same circle, will include equal angles. For if the arcs A F B, and C E D, are equal, the chords A B, C D, are * equal also; and the triangles A O B, C O D, having all their sides respectively equal, * will be equal in all respects. Consequently, the angles A O B, C O D opposite to the equal chords, are equal. * *art.* 97. * *art.* 28.

COR. II.

99. Hence also, the radii which include equal angles in the same circle, will terminate equal arcs. For if the angles A O B, and C O D, are equal; the triangles A O B, C O D, having the sides which include these equal angles * equal, are equal in * all respects. Therefore the chords A B, C D, opposite to the equal angles, are equal, as well as the * arcs A F B, C E D, which they subtend. * *art.* 96. * *art.* 15. * *art.* 97.

COR. III.

100. Lastly, If two diameters A B, C D, are at right angles to each other, they will divide the circumference as well as the circle into four equal parts, called Quadrants. For the arcs and the opposite angles being equal, the sectors will be so too. *Fig.* 3.

THEOREM II.

Fig. 4. 101. *The angles at the center in the same circle are proportional to the arcs terminated by the radii, which include these angles*, viz. AB : BC :: AOB : BOC.

Let the arc BL be any parts $\frac{2}{3}$, of the arc AB; draw the radius OL; then because equal * *art* 99. arcs subtend equal * angles, the angle BOL will be the same parts $\frac{2}{3}$ of the angle AOB, as the arc BL is of AB.

Now if the arc BL, is less than the arc BC, it is plain that the angle BOL is also less than the angle BOC; if the arc BL is equal to the arc BC, the angle BOL is equal to the angle BOC; and if the arc BL is greater than the arc BC, the angle BOL is greater than the angle BOC.

Consequently, since the equal parts BL, BOL, of the antecedents AB, AOB, when compared with the consequents BC, BOC, are both together less, equal to, or greater than the consequents, the angles are proportional to the arcs which subtend them, by defin. 5, of proportion.

COR. I.

102. It is evident, that the sectors AOB, BOC, are also in the same proportion to each other as the arcs AB and BC, which terminate them; for it is easy to perceive that equal arcs terminate equal sectors; and therefore, by the same argument as above, they are proportional to the said arcs.

N. B. Mathematicians suppose the circumference divided into 360 equal parts called Degrees, each degree into 60 equal parts called Minutes

Minutes, and each minute into 60 Seconds. These divisions are commonly marked on a brass semi-circle, or on a rectangular piece of Ivory called Protractor, by which angles are measured or laid down on paper.

COR. II.

103. Hence, the semi-circumference or the measure of two right angles is $\frac{360}{2}$ or 180 degrees; the quadrant or measure of a right angle 90; and lastly, each angle of an equilateral triangle 60; as being the third part of two right angles.

THEOREM III.

104. *If a diameter* KL *bisects any chord* AB, *it will cut it at right angles; and if a diameter is at right angles to a chord, it will bisect it.* Fig. 5.

First, Let H be the intersection; draw the radii OA, OB, then as the triangles OAH, OBH, have the sides OA, OB equal * by the nature of the circle OH common to both, and HA equal to HB, by supposition they are * equal in all respects; and therefore the angles OHA, OHB, opposite to the equal sides OA, OB, are equal, and so are * right angles. * art. 96. * art. 28. * def. 10.

Secondly, If the angles OHA, OHB, are supposed to be right angles; then because in the right angled triangles OHA, OHB, the hypothenuses OA, OB, are equal by the property of the circle, * and the side OH being common to both, they are equal * in all respects. Consequently, the sides AH and HB, opposite to the equal angles at O, are equal. * art. 96. * art. 19.

COR. I.

105. Hence, the line K L which biſects any chord A B at right angles, will paſs through the center O of the circle; or if a line is drawn from the center perpendicular to any chord, it will biſect that chord.

COR. II.

106. If a line drawn through the center perpendicular to a chord be produced to the circumference, it will biſect the arc terminated by that chord; for ſince the angles A O L and B O L are equal, the arcs A L, and L B, as likewiſe their differences A K, B K, to a ſemi-circle, are
* *art.* 99. * equal.

COR. III.

107. It alſo appears, that the arcs A C, B D, terminated by two parallel chords A B, C D, are equal. For the diameter K L which biſects one
* *art.* 7. of theſe chords A B, alſo biſects * the other C D, which is parallel to it, as well as the arcs termina-
* *art.* 99. ted * by theſe chords; therefore the arcs A C, B D, which are the ſum or difference, of equal
* *ax.* 3. arcs * are equal.

THEOREM IV.

Fig. 6. 108. *The chords* A B, C D, *whoſe perpendicular diſtances* O H, O L, *from the center are equal, will be equal; and if the chords are equal their diſtances are ſo.*

Draw the radii, O A, O C, then becauſe O L, O H, are equal, by ſuppoſition, and the hypothenuſes O A, O C, by the property of the circle,
* *art.* 19 the triangles O A H, O L C, are equal in * all reſpects; therefore the ſides A H, C L, or their
* *art.* 104. * doubles A B, C D, are equal.

If

If the chords A B, C D, are equal, their halves A H, C L are * equal, and the triangles O H A, O L C, right angled at H, L, having the sides L C, H A, and O A, O C, equal, are * equal in all respects. Consequently, the sides O H, O L, opposite to the equal angles O A H, O C L, are equal.

* *ax.* 6.

* *art.* 19.

THEOREM V.

109. *The angle at the center* C O D *is double the angle* C A D *at the circumference, which insists on the same arc* C B D. Fig. 7.

Draw the diameter A B and the radii O C, O D; then, because O A, O C, are equal, the triangle A O C is isosceles, and the external angle B O C is * double the internal opposite one B A C; and since O A, O D, are equal, the triangle O A D is isosceles; and the external angle B O D double the internal opposite one B A D. Consequently, the sum C O D, of the external angles, or the angle at the center, is double the sum C A D of the internal opposite ones, or the angle at the circumference, which insists on the same arc.

* *art.* 11.

COR. I.

110. Since the angle at the center is measured * by the arc C B D, its half, C A D, will be measured by half that arc. Consequently, any angle at the circumference is measured by half the arc on which it insists.

* *art.* 101.

COR. II.

111. If the chord A C produced be supposed to turn in the plane of the circle, about the point A as center, till the intersection C coincides with the point A; it is evident, that this line will co-

incide with the tangent E F at A, ſince it touches the circle but in one point: And as the angle B A C is always meaſured by half the arc B C, ſo will the angle B A E be meaſured by half the circumference B C A. Conſequently, the tangent makes a right angle with the diameter drawn thro' the point of contact A.

COR. III.

112. As the angle B A C is meaſured by half the arc B C, and the angle B A E by half the arc B C A, the difference C A E between theſe angles will be meaſured by half the difference C A, of theſe arcs. Conſequently, any angle C A E, made by a chord and a tangent, is meaſured by half the arc terminated by that chord, and ſo is equal to any angle at the circumference which inſiſts on the ſame arc.

COR. IV.

Fig. 8. 113. It is manifeſt, that the angles A C B, A D B, which are contained in the ſame ſegment, or, which is the ſame, which inſiſt on the ſame arc A E B, are equal; as being meaſured by **art.* 110. half * the ſame arc A E B.

COR. V.

114. Hence, each of the angles A C B, A D B, will be leſs, equal to, or greater than a right angle, according as the ſegment in which they are contained is greater, equal to, or leſs than a ſemicircle; ſince half the arc A E B, which is their meaſure, will be leſs, equal to, or greater than a quadrant.

THEOREM IV.

Fig. 9, 10. 115. *The angle contained between two lines drawn from any point* E, *to the circumference, will be*

be measured by half the sum or difference of the two arcs terminated by these lines, according as this point is within or without the circle.

Let A, B, C, D, be the intersections of these lines and the circumference, draw the chord A L parallel to C D, then the angle D E B will be * equal to the angle B A L, and the arc A C * equal to the arc L D. Now as the angle B A L at the circumference is measured by * half the arc L B, on which it stands, and the arc L B being the sum of the arcs B D, and A C, when the point E is within the circle, or their difference, when that point is without: It follows that the angle B E D is measured by half the sum or half the difference of the arcs B D, A C, according as the point E is within or without the circle.

* *def.* 12 *of parallel.*
* *art.* 107.
* *art.* 110.

THEOREM VII.

116. *If two lines* A B, C D, *intersect each other, either within or without the circle, the rectangles of their parts, terminated by the circumference, and their intersection, will be equal,* viz. A E × E B = C E × E D.

Let the lines B C, A D be drawn, then the triangles E A D, E C B, having the angles at B and D equal, as insisting on the same are * A C, and the angle at E common to both, they are * similar: Therefore A E : C E :: E D : E B, or A E × E B = C E × E D.

* *art.* 113.
* *art.* 69.

COR. I.

117. If one of these lines passes through the center, and the other is pendicular to it, as in fig. 5, the diameter K L, to A B, then the rectangle of K H and H L, will be equal to the rect-

Fig. 5,

angle of AH and HB, or because AH is equal to HB; we have $KH \times HL = \overline{AH}^2$. Consequently, any perpendicular to a diameter is a mean proportional, between the parts of that diameter, terminated by the perpendicular.

COR. II.

Fig. 11. 118. Hence, if one of these lines becomes a tangent as EA, the triangles EAD, ECA, will still be similar; since the angles CAE, CDA, **art.* 112. are both measured * by half the arc AC, and the angle E common to both. Consequently, EC: EA :: EA : ED, or $EC \times ED = \overline{EA}^2$.

COR. III.

119. If from the same point E, there be drawn another tangent EB, to the circle, then will $\overline{EB}^2 = EC \times ED = \overline{EA}^2$; or EB = EA; which shews, that if from the same point without the circle there be drawn two tangents, their parts between the points of contact, and their intersection, will be equal.

THEOREM VIII.

Fig. 12, 13. 120. *The greatest chord that can be drawn in a circle is that which passes through the center, and the least that is farthest distant from it.*

From the center O draw the radii OC, OD, to the extremities of any chord CD, not passing through the center; draw likewise the diameter AB; then because the sum of the sides OC, OD, of the triangle OCD is equal to the diameter AB, by the nature * of the circle, and always **art.* 96. greater than the * third side CD; it follows that **art.* 23. the diameter is the greatest chord that can be drawn in a circle.

Now

Now as the line CD recedes from the center O, the angle COD becomes leſs; and conſequently, the * leaſt chord is that which is fartheſt diſtant from the center. * art. 24.

THEOREM IX.

121. *If from any point* E, *either within or without the circle, there be drawn lines to the circumference; that* EB *which paſſes through the center will be the greateſt of all, and the part* EA, *of that line on the other ſide of the center, the leaſt of all.*

Becauſe the ſum of the ſides DO and OE is equal to EB, and always greater than the * other ſide ED: It follows, that the line EB, drawn from any point E, which paſſes through the center, and is terminated by the circle, is the greateſt of all. * art. 23.

And ſince OC is equal to OA, the line AE will be equal to the difference between the ſides EO and OC, and always leſs than the third * ſide EC. Conſequently, the line drawn from any point E, which paſſes through the center when produced, and is terminated by the circumference is the leaſt of all. * art. 23.

THEOREM X.

122. *There can but two lines be drawn from any point* E, *beſides the center, to the circumference, which are equal.*

Let the arcs BD, BF, be equal, draw FG, DC, through the point E, and join BD, BF; then becauſe the arcs BD, BF, as well as DA, FA, are equal by ſuppoſition, the chords BD, BF, are * equal; and as the ſide EB is common to the triangles EBD, EBF, and the angles at B are * equal, they are equal in all * reſpects. * art. 97. * art. 113. * art. 15.

ſpects. Therefore the ſides E D, E F, oppoſite to the equal angles at B, are equal.

But as there can be taken but two arcs B D, B F, from the point B, which are equal, ſo can there neither be made but two triangles, which have the common ſide E B, that are equal in all reſpects. Conſequently, there can be but two lines drawn from the ſame point beſides the center to the circumference, which are equal.

COR. I.

123. Conſequently, if more than two lines can be drawn from any point to the circumference of a circle which are equal, that point will be the center.

COR. II.

124. It is evident, that two circles can cut each other but in two points, becauſe the radii in the ſame circle are equal; the circle deſcribed from the center E can cut the circle B D A F but in two points; ſince there can be drawn but two lines from that point to the circumference, which are equal.

COR. III.

125. It is manifeſt, that two circles will cut each other when the diſtance between their centers is either leſs than the ſum, or greater than the difference of their radii; for if the diſtance between the points E and O is greater than the difference of the ſides E D, O D, there may always be deſcribed two circles from the centers E and O, with the radii E D, O D, which ſhall interſect each other * in the point D; and if the diſtance E O is leſs than the lines E C, O C, the two circles deſcribed from the centers E and O, with the radii

Fig. 13.

* *art.* 23.

Fig. 12.

radii E C, O C, will interſect each other in * the point C. * art. 23.

THEOREM XI.

126. *If from the point* M, *where a tangent touches the circle, a perpendicular be drawn to any diameter* B A, *the radius will be a mean proportional between the parts of that diameter, terminated by the center, that line, and the tangent*, viz. ∴ OP: OA:OT. Fig. 14.

Let the radius OM be drawn; then, becauſe the angle OMT, made by the radius and the tangent, is * a right one, and the angle O common to both the right angled triangles OPM, OMT, they are * equiangular. Therefore * OP: OM::OM:OT, or ∴ OP:OA:OT. * art. 111. * art. 13. * art. 69.

THEOREM XII.

127. *If in a diameter produced there be taken two points* P, T, *ſuch as the radius be a mean proportional, between the diſtances* OP, OT, *of theſe points from the center; the lines drawn from theſe points to any point* M *in the circumference, will be always in a conſtant ratio*, viz. AP:AT::PM: TM. Fig. 15.

In the radius OM, take OQ=OP, and join AQ; then the triangles OQA, OMP, having the angle O common to both, and the adjacent ſides equal, are * equal in all reſpects; that is, AQ=PM; and becauſe OP or OQ:OA:: OM:OT, by ſuppoſition, the line AQ will be * parallel to TM, and therefore * (OQ:OA::) QM or AP:AT::AQ or PM:TM. * art. 15. * art. 68. * art. 67.

THEOREM XIII.

128. *If in a diameter produced there be taken two points* P, T, *ſo as the radius be a mean proportional* Fig. 16 17.

tional between the distances of these points from the center; and if from one of these points a line DT *be drawn at right angles to the diameter; the line* MN, *joining the points of contact of two tangents drawn from any point* D *in that line, will always pass through the other point* P.

From the point D draw a line to the center O intersecting the line MN in Q; then, because
* *art.* 119. * DM is equal to DN, and OM to ON, the triangles OMD, OND, having the side OD common
* *art.* 28. to both, are * equal in all respects.

Hence, the angles MOQ, NOQ, are equal; and since OM, ON, are equal, the line OD
* *art.* 18. * bisects MN at right angles.

Now the right angled triangles OTD, OQP,
* *art.* 13, 69. having the angle at O common to both, are * similar; whence OP : OQ :: OD : OT; but
* *art.* 126. * OQ : OA :: OA : OD; or inversely, OA :
* *art.* 85. OQ :: OD : AN, Consequently, * OP : OA :: OA : OT; and since ∴ OP : OA : OT, by supposition, the line MN passes through the point P.

THEOREM XIV.

Fig. 18. 129. *If two circles intersect each other in two points, the line* OC *joining the centers, will bisect the common chord* AB *at right angles.*

Draw the radii CA, CB, and OA, OB then the triangles OAC, OBC, having all their
* *art.* 28. sides equal, are * equal in all respects; whence the angles ACO, BCO, are equal; and since the sides CA, CB, are equal, the line CO will
* *art.* 18. bisect * the chord AB at right angles.

These are the most material theorems contained in the first, third, fourth, fifth, and sixth books of *Euclid's*

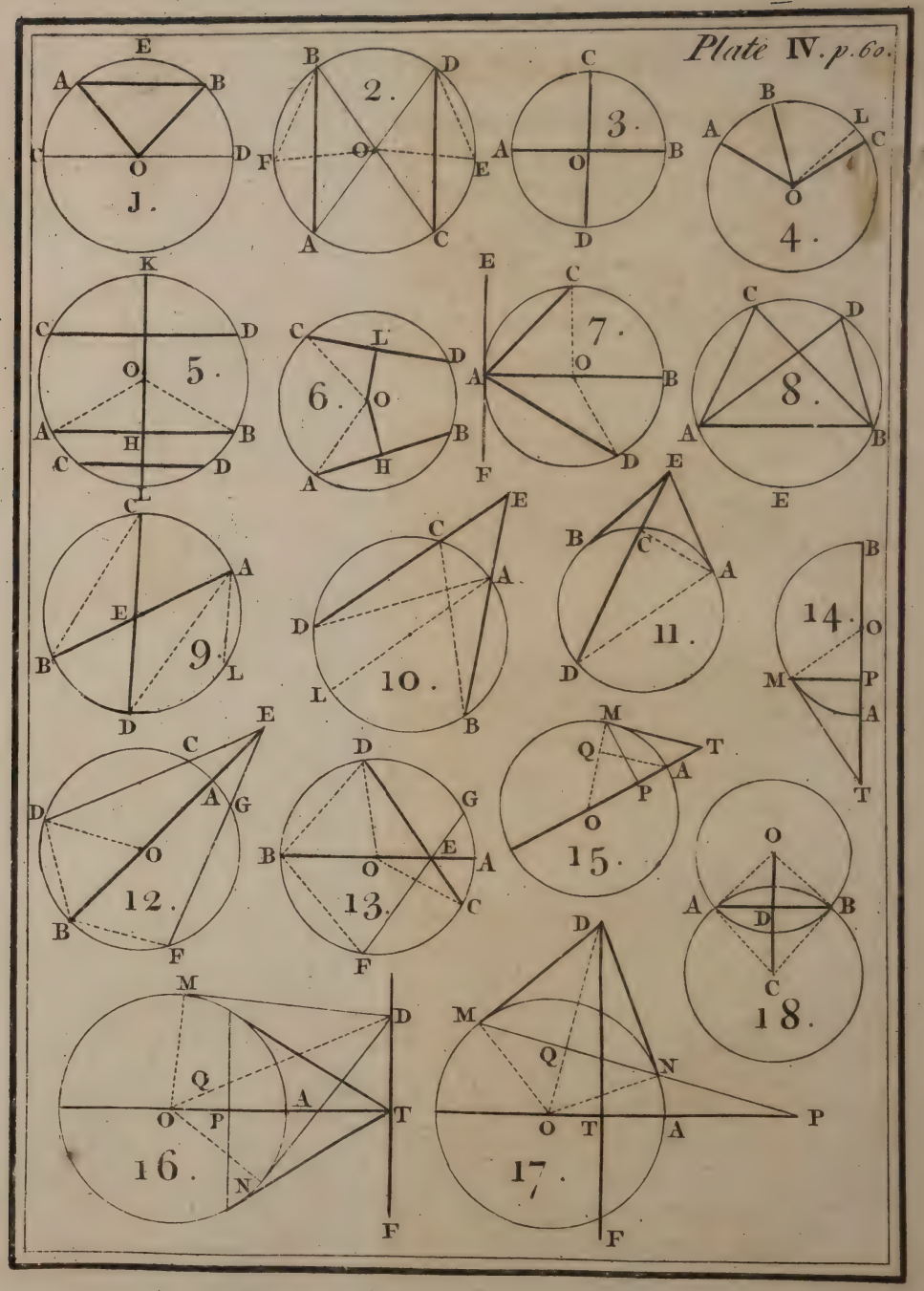
Plate IV. p. 60.
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Euclid's elements, which are useful in the different branches of the mathematics, as far as I ever could find: As to the rest they are only of use to demonstrate some others thereby, for which reason we have omitted them.

As the problems are properly the application of the theorems to geometrical purposes, we thought it more convenient to treat them separately; by which the reader may peruse as many of them as he thinks proper, and leave the rest till he wants them.

SECT. VI.

Of *GEOMETRICAL PROBLEMS.*

PROBLEM I.

130. *THROUGH a given point* C, *to draw a line* CD *parallel to a given line* AB. *Plate V. Fig. 1.*

From the given point C describe an arc EG, so as to cut the given line in E; in AB take EF equal to CE, and from the point F, as center, describe another arc through the point E, meeting the former in G; then the line CG will be parallel to AB.

For, if the lines CE, CF, and FG are drawn, the triangles CEF, CGF, having CF common to both, and the other sides equal by construction, will be equal in all * respects. Hence, the alternate angles CFE, FCG, opposite to the equal sides CE, FG, are equal. Consequently, CD is * parallel to AB.

* *art.* 28.

* *art.* 6.

PROB.

PROB. II.

Fig. 2. 131. *From a given point* C, *either within or without a given line* AB, *to draw a perpendicular to that line.*

From the given point C, as center, describe an arc so as to meet the line AB in two points E, F; and from these points, as centers, with the same radius greater than half of EF, describe two arcs so as to meet in D; then the line CD will be perpendicular to AB.

For if FD, ED; FC, EC, are drawn, the triangles CED, CFD, having CD common to both, and the other sides equal by construction, * *art.* 28. will be equal * in all respects; whence the angles EDC, FDC, are equal; and since ED, FD, * *art.* 18. are equal, the line DC is perpendicular to AB.

PROB. III.

Fig. 3. 132. *At the extremity of a given line* AB *to erect a perpendicular* BD *to that line.*

From any point O without the given line as center, describe an arc through the given point B, and so as to cut the line AB in some other point E; draw the diameter ED; then the line drawn through the extremity D to the given point B will be perpendicular to AB.

This is evident, since the angle EBD contained * *art.* 114. in a semi-circle is * a right angle.

PROB. IV.

Fig. 4. 133. *To bisect a given line* AB *by a perpendicular* DE.

From the extremities A, B, of the given line as centers, describe arcs on both sides, with any radius, so as to intersect each other at D and E; then

then the line DE drawn through these intersections, will bisect AB at right angles.

Let the points A, B, be joined to the points D, E, then the triangles ADE, BDE, having DE common to both, and the other sides equal by construction they are * equal in all respects: And hence the angles ADE, BDE, opposite to the equal sides AE, BE, are equal; and since AD and BD are equal, the line DE * bisects AB at right angles. * *art.* 28. * *art.* 18.

PROB. V.

134. *To bisect any given angle* ACB. *Fig.* 5.

From the angular point C as center, with any radius, describe an arc meeting the sides in A, B, and from these points as centers, describe two arcs, with any but the same radius, so as to meet in D; then the line DC will bisect the angle ACB.

For since DC is common to both triangles, ACD, BCD, CA equal to CB, and BD to AD by construction, they will be equal * in all respects. Therefore the angles ACD, BCD, opposite to the equal sides AD, BD, are equal. * *art.* 28.

PROB. VI.

135. *With a given line* AC *to make an angle equal to a given one acb.* *Fig.* 6.

From the points C, *c*, as centers describe two arcs with any but the same radius; take the arc AB equal to the arc *ab*; then the line CB will, with CA, make the angle ACB equal to the given one *acb*.

This is evident, since the radii CA, *ca*, are equal, as well as the arcs AB, *ab*.

PROB.

PROB. VII.

Fig. 7. 136. *To describe a triangle* ABC, *whose sides shall be equal to three given lines* AC, DE, FG.

From the extremity A of one of these lines as center, describe an arc, with a radius equal to DE, and from the other extremity C as center describe another arc with a radius equal to FG, so as to meet the first in B; by drawing AB, CB, you will have the triangle required.

For since all the radii of the same circle are equal, the side AB is equal to the line DE, and the side CB to the line FG.

PROB. VIII.

Fig. 8. 137. *To describe a rectangle* AC, *whose sides shall be equal to two given lines* AB, EF.

At the extremities A, B, of one of these lines * *art.* 132. * erect two perpendiculars AD, BC, each equal to the other line EF, and draw DC; then will ABCD be the rectangle, whose sides are equal to the given lines AB, EF.

This is evident from the construction; since AD, BC, are parallel and equal, DC will be parallel and equal to AB.

PROB. IX.

Fig. 9. 138. *To describe a parallelogram* DF, *equal to a given triangle* ABC, *whose sides shall make a given angle.*

Through the vertex B, of the given triangle, * *art.* 130. draw the line BF parallel to the base CA; bisect * *art.* 133. * the base AC in D, and draw DE so as to make with AC the given angle; then, by drawing AF parallel to DE, the parallelogram DF will be equal to the given triangle ABC.

* *art.* 35. For since a triangle is half * the parallelogram of

of the same base and altitude, the triangle A B C, whose base is double the base of the parallelogram D F, and has the same altitude, will consequently be equal to it; and the angle C D E is equal to the given angle by construction.

PROB. X.

139. *On a given line to describe a parallelogram* E F, *equal to a given triangle* A B C, *which shall have a given angle.* Fig. 10.

Through the vertex B of the given triangle, draw a line B G parallel to the base A C, bisect the base in D, draw D I so as to make the given angle with D C; produce D C to H so as C H be equal to the given line; draw C K, H F, parallel to D I, and through the point H and the intersection of B G and C K, draw H E so as to meet D I in I; then if I F be drawn parallel to B G, meeting C K, H F, in K and F, the parallelogram E G F K will be the required figure.

For the parallelogram D E is *equal to the triangle A B C, as likewise to its complement * E F. Therefore the parallelogram E F is equal to the given triangle A B C, the side E G equal to the given line by construction, and the angle G E K equal to the given angle. * art. 138 * art. 38.

PROB. XI.

140. *To make a rectangle* G F, *equal to any given trapezium* A B C D. Fig. 11.

Bisect the sides A B, B C, by the line E F, and the sides D A, D C, by G H; and through the angles A, C, draw lines perpendicular to E F; then will E F H G be the rectangle required.

Draw the diagonal A C, then it is evident * that E F, G H, are both parallel to A C, and so parallel to each other, whence G F is a rectangle; but * art. 68.

the rectangle AF having the same base AC, and
* *art.* 35 half the altitude of the triangle ABC, is * equal to it; and the rectangle AH, having the same base, and half the altitude of the triangle ADC, they are likewise equal. Consequently, the rectangle GF is equal to the trapezium ABCD.

PROB. XII.

Fig. 12. 141. *To make a triangle* ABC, *similar to a given triangle* DEF, *which shall have a given side* AC.

* *art.* 135. At the extremity A of the given line, * draw the line AB, so as the angle A be equal to the angle D, and at the other extremity C, the line CB, so as the angle C be equal to the angle F; then the triangle ABC will be * equiangular and
* *art.* 13. * similar to the triangle DEF, and the side AC
* *art.* 69. equal to the given line.

COR.

142. It is evident, that any rectilinear figure may be made similar to any other given figure, by reducing it into triangles, since there is no more required than to make upon a given line as many similar triangles as the given figure has been divided into; and therefore the triangles being similar in both, the figures themselves will be * si-
* *art.* 94 milar.

PROB. XIII.

Fig. 13. 143. *To divide a given line* AB *into any number of equal parts as* 5.

Through the extremities A, B, draw the indefinite lines AC, BD, parallel to each other, making any angle with AB; then if from the point A towards C, there be set off as many equal parts less one, on AC, as the line AB is to be divided into,

into, and the same number of parts equal to the former, are set off on BD, from B towards D; the lines which join the opposite points, will divide the given line into the desired number of equal parts.

For, because the lines 43, 32, are equal and parallel to 12, 23, the lines 14, 23, are * likewise parallel; and therefore A1 : 12 :: AE : EF, *&c.* and since A1 is equal to 12, AE will also be equal to EF, and whatever part A1 is of A4, AE will be the same part less one of AB. * *art.* 31.

PROB. XIV.

144. *To find a fourth proportional to three given lines a, b, c.* *Fig.* 14.

Draw AK and AL so as to make any angle, on AK; take AB = *a*, BC = *b*, and on AL take AD = *c*; then if BD be drawn, and CE parallel to it, DE will be the fourth proportional required, since * AB : BC :: AD : DE. * *art.* 67.

PROB. XV.

145. *To find a third continued proportional to two given lines a, b.*

On AK take AB = *a*, BC = *b*, and on AL take AD = *b*, join BD, and draw CE parallel to BD; then will DE be the third proportional required. For AB *(a)* : BC *(b)* :: AD *(b)* : DE.

PROB. XVI.

146. *To find a mean proportional between two given lines a, b.* *Fig.* 15.

Draw the line AB equal to the sum of the two given lines, upon AB, as a diameter describe the semi-circumference of a circle AMB; take AP equal to one of these lines *a*, and erect PM perpendicular to the diameter AB, which being

terminated by the circumference, will be the mean proportional required.

If the chords AM, BM, are drawn, the angle AMB, being contained in a semi-circle, will be a right angle; and the right angled triangles APM, BPM, having each an angle A, B, com-
* *art.* 13. mon with the triangle AMB, will be * equiangu-
* *art.* 69. lar and similar. Therefore* AP : PM :: PM : PB.

PROB. XVII.

Fig. 16. 147. *To expreſs the ratio of any two rectilinear figures* P, Q, *by two lines; that is, to find two lines which have the ſame ratio to each other as any two rectilinear figures.*

If they are parallelograms or rectangles, let ABCD be equal to P; make the parallelogram
* *art.* 139. or rectangle DE equal * to Q, whoſe ſide FE is
* *art.* 65. given; then *will AC : DE :: P : Q : AD : DF.

If the figures P, Q, are trapeziums, you may find rectangles equal to them by *prob.* 11; and if they are any polygons, they may be reduced into triangles, and the triangles into rectangles.

PROB. XVIII.

Plate VI. *Fig.* 17. 148. *To deſcribe the circumference of a circle through three given points* A, B, C, *not placed in a right line.*

Join theſe points by two lines AB, AC;
* *art.* 133. which biſect * by the perpendiculars DO, EO; and from the interſection O of theſe perpendiculars, as center, deſcribe the circumference through the point A, which will be the required one.

For ſince OD biſects the chord AB at right angles, the circumference deſcribed through the
* *art.* 105. point A will * paſs through the point B; and becauſe OE biſects the chord AC at right angles, the circumference which paſſes through the point

A

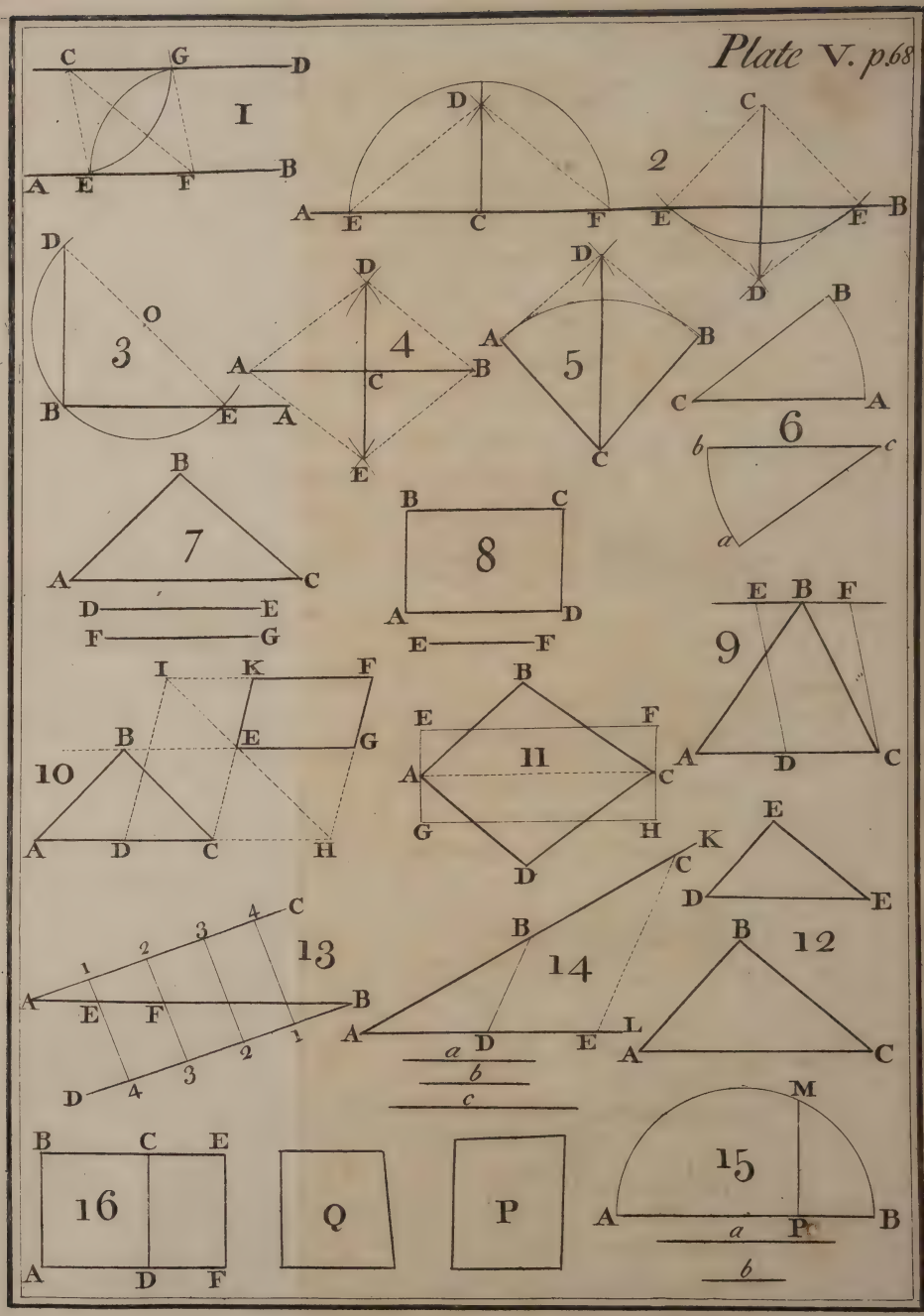
Plate V. p.68
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Q
P

A will also pass through the point C. Therefore O is the center of the circle required.

PROB. XIX.

149. *From a given point* T *without a circle to draw a tangent to that circle.* Fig. 18.

Join the center O to the given point T; on OT, as diameter, describe a semi-circumference intersecting the former in M; then the line drawn through this intersection to the point T, will be the tangent required.

For if the radius OM be drawn, the angle OMT contained in a semi-circle will be a * right angle. Therefore the line MT touches the circle * at the point M. * art. 114. * art. 111.

If it was required to draw a tangent from a given point M in the circumference, it is evident that the line, erected perpendicular at the extremity M, to the radius OM, will be the tangent required.

PROB. XX.

150. *To inscribe a triangle* ABC *in a circle, similar to a given triangle abc.* Fig. 19.

Draw the tangent DE to any point B, and from the point B draw the chord BA so * as the angle DBA be equal to the angle *c* of the given triangle; and BC so as the angle EBC be equal to the angle *a*, join AC; then will ABC be the triangle required. * art. 135.

For the angles ACB, ABD, are equal * by the nature of the circle; and because the angle ABD has been made equal to the angle *c*, the angles ACB, *c*, * are equal; the angles BAC, CBE, are * also equal: And since the angle CBE has been made equal to the angle *a*, the angles BAC, *a*, * are equal. Consequently, the triangle ABC, is * equiangular and * similar to the triangle *abc*. * art. 112. * ax. 1. * art. 112. * ax. 1. * art. 13.

PROB. XXI.

Fig. 20. 151. *About a circle to describe a triangle* DEF, *similar to a given triangle abc.*

Produce the base *ac* of the given triangle to *m*, *n*; draw the radii OA, OB, so as to make an angle equal to the external angle *mab*; draw likewise the radius OC, so as the angle BOC be equal to the external angle *ncb*; then the lines which touch the circle in A, B, C, will form the triangles DEF, similar to the triangle *abc*.

* *art.* 111. For the angles at A and B are * right angles, and the angle AOB equal to the angle *mab*, by construction; the angle D, which makes two * *art.* 32. * right angles with the angle AOB, will be equal to the angle *bac*, which makes with the an- * *ax.* 1. gle *mab*, equal to the angle AOB, * likewise two right angles.

It is proved in the same manner, that the angle E, which makes two right angles with BOC, is equal to the angle *bca*, which makes two right angles with its equal *ncb*. Therefore the trian- * *art.* 13. gle DEF is equiangular * and similar * to the * *art.* 69. triangle *abc*.

PROB. XXII.

Fig. 21. 152. *To describe a circle which shall touch the three sides of a given triangle* ABC.

* *art.* 134. Bisect * any two angles B, C, by the lines BO, * *art.* 131. CO, and from their intersection O, draw * OD perpendicular to AB; then the circle described from the center O with the radius OD, will touch the three sides of the triangle ABC.

From the center O draw OE, OF, perpendicular to BC and AC; then because the right angled triangles BDO, BEO, having the side BO common

common to both, and the angles at B equal by conſtruction, are * equal in all reſpects; therefore OD, OE, oppoſite to the equal angles at B are equal. * art. 25.

And the right angled triangles CEO, CFO, having the ſide CO common to both, and the angles at C equal by conſtruction, are alſo * equal in all reſpects; hence OE and OF are equal. Conſequently, ſince the three lines OD, OE, OF, are equal and perpendicular to the ſides of the triangle, the circle will touch * the three ſides of that triangle. * art. 25. * art. 111.

PROB. XXIII.

153. *On a given line* AB *as a chord to deſcribe a ſegment of a circle, ſo as to contain a given angle.* Fig. 22.

Draw DE ſuch that the angle DAB be equal to the given angle; let AC be perpendicular to AD, and BC to AB, meeting the former in C; then the circle deſcribed on the diameter AC, will give the ſegment AGCB required.

For CAD, ABC, are right angles by conſtruction, therefore the circumference * paſſes thro' the point B, and touches the * line DE; and ſince the angle DAB has been made equal to the given angle, and is equal to any angle contained in the * ſegment AGCB, it will be that required. * art. 114. * art. 111. * art. 112.

PROB. XXIV.

154. *The angles* AOB, COB, *under which three objects* A, B, C, *placed in a right line, whoſe diſtances are known, are ſeen from any place* O, *being given; to find the diſtances from the place* O *to theſe objects.* Fig. 23.

Make the angles DCA, DAC, each equal * to their alternate given angles AOB, COB; thro' the point D and the extremes A, C, deſcribe * a * art. 135. * art. 148.

circumference of a circle; then the line DB produced, so as to meet the circumference, will determine the point O.

For, the angles AOD, DCA, which insist on
* *art.* 113. the same arc AD, * are equal, as well as the angles BOC and DAC. Therefore the construction is rightly performed.

PROB. XXV.

Fig. 24, 25. 155. *The distances between three objects* A, B, C, *being given, to find a point* D, *such as the lines drawn from it to the three objects shall contain given angles.*

* *art.* 153. On the side AB as a chord, describe a * segment of a circle, so as to contain one of these given angles; describe likewise on AC as a chord, a segment so as to contain another of these given angles; then the intersection D of these arcs will be the point sought. This is evident by the construction.

N. B. When the point sought D is without the triangle, it is plain that if the angle ACB is less, equal to, or greater than the given angle ADB, the angle ABC must also be less, equal to, or greater than the given angle ADC, otherwise the problem is impossible; and when the angle BDC is equal to the sum of the angles ABC, BCA, the four points A, B, C, D, fall in the same circumference, and consequently the problem is indetermined in this case.

PROB. XXVI.

Fig. 26, 27. 156. *To describe the circumference of a circle through two given points* C, G, *so as the segment cut off by a line* EF *given in position, shall contain a given angle.*

Bisect

Biſect in L the line joining the given points by the perpendicular OL, which meets the line EF in H, draw GH, and from the interſection L the line LB, ſo as the angle LBH be equal to the difference between the given angle and a right one; then if LD is made equal to LB, and GO drawn parallel to DL, meeting HL in O, this point O will be the center of the circle required. For becauſe OL biſects CG, at right angles, the circumference paſſing through one of theſe points will alſo paſs * through the other. * art. 105.

If OE be drawn parallel to LB, then * ſince * art. 67. HL:HO::LD:OG::LB:OE; and as LD=LB, by conſtruction, OE will be equal to OG; and ſo the point E is in the circumference.

If EO be produced to the circumference K, then becauſe any angle in the ſegment ECF is meaſured by half the arc EGF, and the angle KEF by half the arc FK, that is, by the difference between the arc EGF, and the ſemi-circumference EGK; it follows that the angle OEH, or its equal LBH, is the difference between the given angle and a right angle.

N. B. Firſt, If the line LB can be carried from the point L to any two different points in the line HG, produced either way, the problem will have two ſolutions; but if it cannot be carried to any point in that line it is impoſſible.

Secondly, If the given angle be a right one, the center O will coincide with the point H. For EF will be a diameter of the circle, as well as the line HL produced; and therefore their interſection H will be the center.

Thirdly, If the line CG interſects the line EF at right angles, the line LO will be parallel to EF;

EF; and hence LO = BE, and LB = OE; that is, LB will then be the radius. Therefore the two circular arcs described from the centers C, G, with the radius BL, will intersect each other in the center O.

Fig. 26. *Fourthly*, If the point L coincides with the point H; that is, if EF bisects CG, then GH will coincide with LG; whence, if from any point in HO, excepting the point L, the line LB be drawn as before, and carried from that point somewhere on CG, and the line GO parallel to this line will give the center O.

Fig. 27. *Fifthly*, And if C, G, coincide, so that the line CG becomes a tangent, and L the point of contact, then will HG become = HL, OL = OG, HD = HL + (LD =) LB. Therefore the proportion HD : HG :: LD : OG, will become in this case HL + LB : HL :: LB : LO, by which the center O may be found.

COR.

Fig. 28. 157. Hence, two points E, F, may be found in a right line given in position, from which lines being drawn to two other given points R, C, without that line, shall make a right angle ERF on one side, and any other given angle ECF on the other. For if RA be drawn perpendicular to EF, and in CA, produced, there be taken the point G, so as ∴ CA : AR : AG, the circumference of a circle described through the points C, G, and such that the segment ECF may contain the given angle, will intersect FF in the points required.

For by the property of the circle we have
* art. 116. * EA × AF = CA × AG, and CA × AG = $\overline{AR}^2$, by construction; therefore EA × AF = $\overline{AR}^2$, or ∴ EA : AR : AF. Consequently, the angle
* art. 146. ERF is a * right angle. This

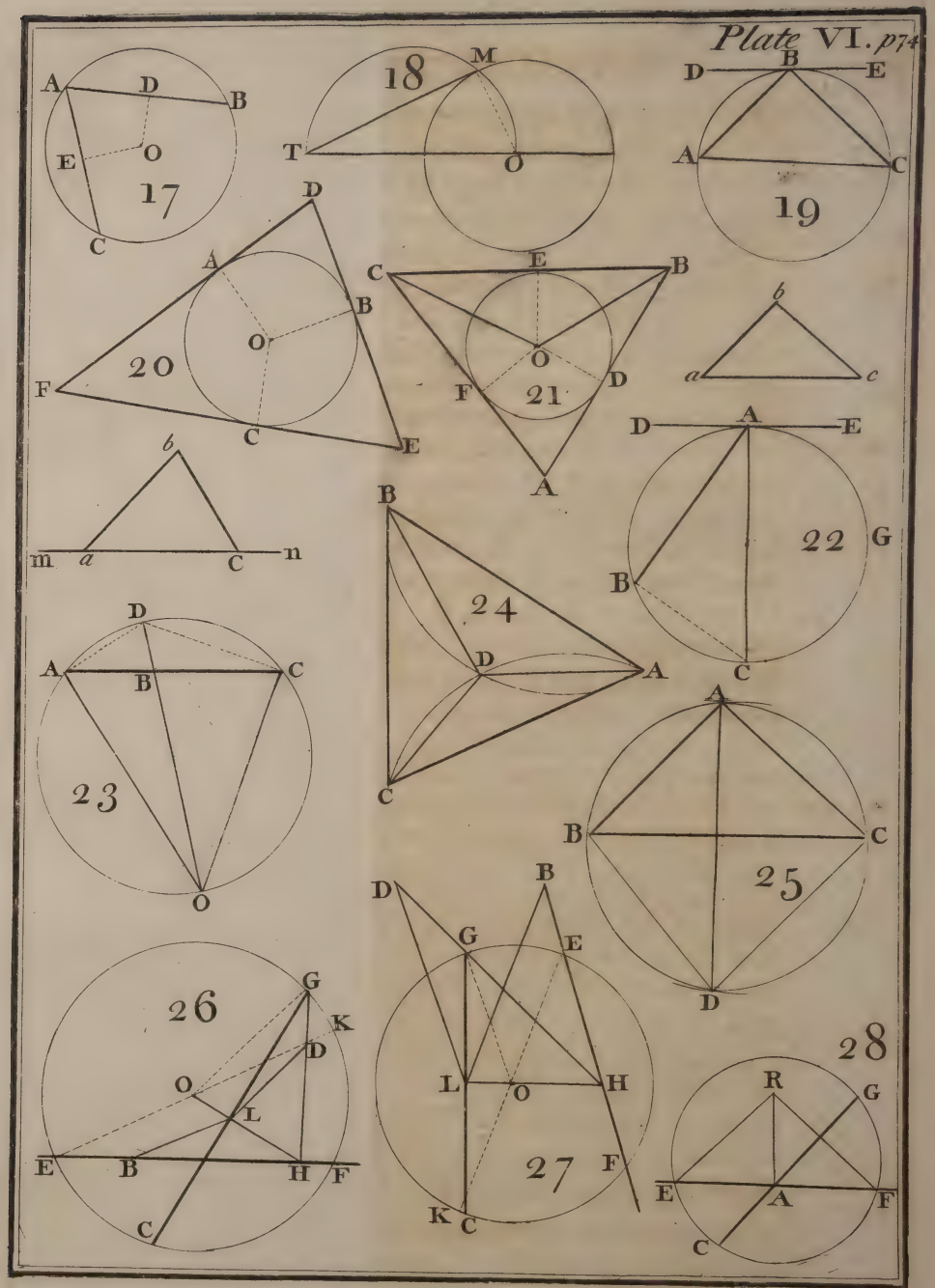
Plate VI. p74
17
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K
C
F
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R
G
E
A
F
C

This problem is useful in conic-sections, to find two conjugate diameters of an ellipsis or hyperbola, which shall make a given angle, when any two conjugate diameters are given.

SECT. VII.

Of *REGULAR POLYGONS.*

DEFINITIONS.

1. WHEN a plane figure is termined by more than four sides, it is called a Polygon.

2. When all the sides are equal as well as all the angles, the polygon is said to be Regular; and Irregular, if the sides or angles are unequal.

3. A polygon is said to be Inscribed in a circle, when all the angles touch, or are in the circumference.

4. And Circumscribed, when all its sides touch the circle.

5. The Angle at the Center of a polygon, is that which is made by two radii drawn to the extremities of the same side.

6. The Angle of the Polygon, is that which is made by any two contiguous sides.

7. Polygons are distinguished according to the number of their sides.

Number of sides,		Number of sides,	
5.	Pentagon.	9.	Enneagon.
6.	Exagon.	10.	Decagon.
7.	Eptagon.	11.	Ondecagon.
8.	Octagon.	12.	Dodecagon.

THE-

THEOREM I.

Plate VII. *Fig.* 1. 158. *The angle* AOB, *at the center of a regular polygon, is equal to* 360 *degrees, or four right angles divided by the number of its sides.*

For all the angles at the center taken together are equal to four * right angles; that is, to 360 degrees, and their number is equal to the number of the sides, because if the radii are drawn to the extremities of all the sides, the polygon will contain as many equal isosceles triangles as there are sides: And therefore the sum 360 of all the angles at the center divided by their number will be equal to the angle at the center required.

* *art.* 4.

THEOREM II.

159. *The angle of a regular polygon is equal to the difference between two right angles and the angle at the center.*

For since AB and BC are equal by supposition, the isosceles triangles OAB, OBC, are *equal in all respects; and hence the angles OAB, OBC, are equal. Therefore the angle ABC of the polygon, is equal to the sum of the angles OAB, OBA, of the triangle OAB; and since the three angles of a triangle are * equal to two right angles, the angle ABC of the polygon will be equal to the difference between two right angles, and the angle AOB at the center.

* *art.* 28.

* *art.* 12.

COR.

COR.

160. Hence, if n expreſſes the number of the ſides of any regular polygon, then * will $\frac{4\,4\,2}{n}$ expreſs the angle at the center of that polygon, and $180--\frac{4\,4\,2}{n}$ or its equal $180\times\frac{n-2}{n}$ the * angle of the polygon. Conſequently, * art. 158. * art. 160.

Polygon.	*Ang. Cent.*	*Ang. Polyg.*
5	72	108
6	63	120
7	$51\frac{1}{7}$	$128\frac{4}{7}$
8	45	135
9	40	140
10	36	144
11	$32\frac{8}{11}$	$147\frac{3}{11}$
12	30	150

THEOREM III.

161. *If a Decagon be inſcribed in a circle, the radius will be a mean proportional between the ſum of the radius and the ſide of the decagon, and that ſide.* Fig. 2.

Let the arcs AB, BC, be each (36 degrees) the tenth part of the circumference; if the ſide AB, and the radius OC, both produced, meet in D. Then becauſe AB is the tenth part of the whole circumference, it will be the fifth of ACE; and therefore the arc BCE four fifths, and half that arc, or the meaſure of the angle OAB, two fifths, 72; that is, double the angle AOB at the center, or equal to the angle AOD; therefore the triangle ADO is iſoſceles, and ſimilar to the triangle * AOB. Hence, the angle ODA is equal to the angle AOB or BOD by ſuppoſition; ſo the * art. 13. 69.

the triangle OBD, having two equal angles O, D,
* *art.* 26. is likewise * isosceles. Whence BD = BO, and AD is equal to the sum of the side AB of the decagon and the radius BO.

But because of the similar triangles ADO,
* *art.* 69. OAB, we have * AB : AO :: AO : AD or AB + BO. Consequently, the radius is a mean proportional between the sum of the side of the decagon and radius, and the side of the decagon.

THEOREM IV.

Fig. 3. 162. *The sum of the squares of the radius and the side of a decagon inscribed in a circle, will be equal to the square of the side of a pentagon inscribed in the same circle.*

Let AB be the side of the Pentagon, bisect the arc AB in C, then will BC or AC be the side of the decagon; bisect the arc AC in E, by the radius OE, and from its intersection D, with AB draw DC.

Then because the arcs AC, CB, and AE, EC, are equal by construction, the arc EB will be 54 degrees, the three fourth of the arc ACB;
* *art.* 160 and therefore the angle EOB is * the three fourths, 54 of 72, the angle AOB.
* *art.* 158. But the angle at the center AOB is * two tenths 72 of four right angles, or four tenths of two right angles; and as the three angles of a triangle
* *art.* 12. AOB are equal * to two right angles, each of the equal angles A or B will be three tenths of two right angles; that is, the angle OBA is the three fourths, 54, of the angle AOB, and so equal to the
* *art.* 26. angle EOB. Consequently, the triangle OBD * is isosceles and similar to the triangle AOB, as ha-
* *art.* 20. ving the common angle * B. So then * AB :
* *art.* 69. BO :: BO : BD, or * AB × BD = BO^2.
* *art.* 73.

And

And since the radius OE bisects the arc AC, it also * bisects its chord AC at right angles; and the lines DA and DC * are equal. Whence the isosceles triangles ADC, ACB, having the angle A common to both * are similar, and * AB: AC :: AC : AD; or * $AB \times AD = \overline{AC}^2$.

* art. 106.
* art. 15.
* art. 20.
* art. 69.
* art. 73.

By adding the former equality to the latter, then $AB \times BD + AB \times AD = \overline{AC}^2 + \overline{BO}^2$; but * $AB \times BD + AB \times AD = \overline{AB}^2$. Consequently, $\overline{AC}^2 + \overline{BO}^2 = \overline{AB}^2$.

* art. 48.

PROB. I.

163. *To inscribe a square in a given circle.* Fig. 4.

Draw a diameter AB and another CD at right angles to it; then the lines joining their extremities will form the square ACBD required.

For the sides being the chords of equal arcs * are equal, and each of the angles being contained in a semi-circle will * be a right angle.

* art. 97.
* art. 114.

If the arc AC be bisected in E, the chord AE or EC of half that arc will be the side of an octagon inscribed in the same circle.

It may be observed, that whatever the number of the sides of a polygon inscribed in a circle may be, you may always find the side of a polygon double the number of the sides, but not triple, nor any other unequal part, at least not by common geometry.

PROB. II.

164. *To inscribe a pentagon or a decagon in a given circle.* Fig. 5.

Draw the radius OD at right angles to the diameter AB; bisect the radius AO in C, and from the point C, as center, describe an arc through the point D, meeting AB in E; then OE will be the side

ſide of a decagon, and the line drawn from D to E the ſide of a pentagon inſcribed in the ſame circle.

Let the arc ED be continued ſo as to meet the diameter BA, produced in F; then becauſe AC, CO, and CF, CE, are equal by conſtruction, AF will be equal to OE, and FO equal to AE.

*art. 117. But by the property of the circle we have * FO or AE : OD :: OD : OE. Since therefore the radius OD is a mean proportional between the ſum of the radius AO, and the line OE, and that

*art. 161. line; OE will be the *ſide of the decagon inſcribed in that circle. And becauſe of the right angled

* art. 49. triangle DOE, the * ſquare of DE is equal to the ſum of the ſquares of OD and OE, the line DE

*art. 162. will *be the ſide of the pentagon inſcribed in that circle.

When a line is divided into two ſuch parts, that the greater part is a mean proportional between the whole line and the leſſer part, that line is ſaid to be divided into mean and extreme proportion.

Hence, the radius OB is divided into mean and extreme proportion in E. For ſince AE : OB :: OB : OE, by ſubtracting the conſequents from their antecedents, and comparing the differences with the conſequents, we have OE : OB :: BE : OE, or inverſely, OB : OE :: OE : BE.

PROB. III.

Fig 6. 165. *To inſcribe an exagon in a given circle.*

If the radius AO be carried round the circumference, you will have the exagon required.

For the angle at the center O is the ſixth part of four right angles, or one third of two right angles; therefore each of the equal angles A, or B, will likewiſe be a third part of two right angles. Conſequently, the triangle OAB is equilateral.

If

If the arc AB be biſected in E, the chord of AE or EB will be the ſide of a dodecagon, inſcribed in that circle.

PROB. IV.

166. *To deſcribe a polygon about a given circle of any number of ſides.* Fig. 7.

Find the ſide AB of a polygon inſcribed in the circle of the ſame number of ſides, draw the radii OA, OB, through its extremities, then the tangent DE parallel to AB, terminated by theſe radii produced, will be one of the ſides of the polygon required, and the reſt may be found in the ſame manner; or elſe deſcribe a circle paſſing through the points D, E, then the ſide already found being carried round this laſt circumference, will touch the inner circle, and ſo will be the polygon required. This is ſelf evident.

PROB. V.

167. *To find the radius of a circle, in which a polygon whoſe ſide is given may be inſcribed.* Fig. 8.

Deſcribe any circle at pleaſure, in which find the ſide AB of a polygon ſimilar to the required one; in AB, produced if neceſſary, take AD equal to the given ſide; draw DC parallel to the radius BO; then CD or CA will be the radius required.

For the triangle CAD is ſimilar to * the iſoſceles triangle OAB; and therefore the angle ACD equal to the angle at the center AOB, of the polygon required, and CA or CD will be the radius ſought. * art 67.

Theſe and other ſuch like problems are eaſily ſolved by means of a ſector, on the inſide edge of which are marked the lines of polygons, from four to twelve ſides: When the ſector is ſo opened, as the interval from 6 to 6 be equal to the radius of any circle,

cle, the distances between the respective numbers expressing the number of sides of the polygons, will be the sides of these polygons inscribed in that circle; viz. from 7 to 7, the side of an eptagon; from 8 to 8, the side of an octagon, &c.

On the contrary, if the sector is so opened as the interval between the respective numbers be equal to the given side of the polygon, then the distance from 6 to 6 will be the radius of the circle, in which that polygon may be inscribed.

THEOREM V.

Fig. 9. 168. *Similar polygons, inscribed in, or circumscribed about, circles, are proportional to the squares of their radii or diameters.*

Let the radii be drawn to the extremities of the sides, then each polygon will be divided into as many equal triangles as there are sides, and those in the one will be similar to those in the other, as
* art. 20. being isosceles, * and having the angles at the
* art. 69. center equal. Therefore * $AOB : HQI :: \overline{AO}^2 : \overline{HQ}^2$; and if n expresses the number of the sides,
* art. 55 * $nAOB : nHQI :: \overline{AO}^2 : \overline{HQ}^2$; that is, the polygon in the one is to the similar polygon in the other, as the square of the radius in the first to the
* art. 55. square of the radius in the second; or as * the square of the diameter in the first is to the square of the diameter in the second.

The similar circumscribed polygons are in the same proportion.

THEOREM VI.

169. *Similar inscribed or circumscribed polygons have the sum of their sides proportional to the radii or diameters.*

Let

Let n express the number of their sides, then because of the similar triangles OAB, QHI, we have * AB : HI :: AO : HQ; or, * nAB : nHI :: AO : HQ :: * AD : HL. * art. 69. * art. 55. * art. 55.

The same thing may be proved in respect to the circumscribed polygons. Consequently, similar polygons inscribed or circumscribed have the sum of their sides proportional to the radii or diameters.

THEOREM VII.

170. *The areas of circles are proportional to the squares of their radii or diameters. I say,* $\overline{AO}^2$: ABCDF :: $\overline{HQ}^2$: HIKLN.

Inscribe and circumscribe similar polygons; then because * $\overline{AO}^2$: nAOB : $\overline{HQ}^2$: nHQI; and we have proved that * equal parts where proportional to the whole's; it is evident, that similar inscribed, or circumscribed polygons, are equal parts of the squares of the radii or diameters. * art. 55. * art. 55.

Now because, if the parts nAOB of the first antecedent $\overline{AO}_2$, are less than the circle or its consequent; the parts nHQI of the second antecedent $\overline{HQ}^2$, will also be less than the circle or its consequent: And if the parts nAOB are greater than its consequent, the parts nHQI are also greater than the consequent; and if possible, that the parts nAOB are equal to the circle, the parts nHQI must also be equal to the circle; for if one was either greater or less, we have proved that the other must likewise be greater or less. Consequently, since the equal parts of the antecedents, when compared to their respective consequents, are both together less, equal to, or greater than the consequents, these quantities are proportional

by defin. 5, of proportions. Whence the areas of circles are as the ſquares of their radii or diameters.

THEOREM. VIII.

171. *The circumferences of circles are proportional to their radii or diameters. I ſay,* AO: ABCDEG::HQ:HIKLMN.

Let *n* be the number of the ſides of ſimilar inſcribed or circumſcribed polygons; then as * AO: *n*AB::HQ:*n*HI, it is evident, by what has been ſaid in the laſt propoſition that the ſums *n*AB, *n*HI, of the ſides of the inſcribed or circumſcribed polygons, are equal parts of the radii or diameters: Now it is plain, that if the parts *n*AB are leſs than the circumference ACE, the parts *n*HI are alſo leſs than the circumference HKM; if the parts *n*AB, are greater than ACE, the parts *n*HI are likewiſe greater than HKM, and if poſſible that *n*AB is equal to ACE, the parts *n*HI muſt alſo be equal to HKM: Since if one is either greater or leſs, the other is alſo greater or leſs. Therefore, ſince the equal parts of the antecedents, when compared with the reſpective conſequents, are both together leſs, equal to, or greater than the conſequents; theſe quantities are proportional by defin. 5, of proportion. Conſequently, the circumferences of circles are as their radii or diameters.

* *art.* 55.

COR.

172. It is evident, that whatever has been demonſtrated in regard to the whole circles, is likewiſe true in regard to their ſimilar parts; that is, the ſector AOB is to the ſimilar ſector HQI, as the ſquare of AO to the ſquare of HQ; and any arc AB is to the ſimilar arc HI, as AO to HQ. By ſimilar arcs and ſectors we underſtand thoſe which

which are terminated by radii that make equal angles at the center. This is evident, ſince the whole's are proportional to their equal parts.

THEOREM IX.

173. *The area of a circle is equal to a triangle* OFL, *whoſe baſe* FL, *is equal to the circumference, and altitude equal to the radius.*

As the area of any polygon, is equal to a triangle whoſe baſe is equal to the ſum of its ſides, and whoſe altitude, the line drawn from the center perpendicular to one of its ſides; and ſince the area of any inſcribed polygon is always leſs than that of the circle, as alſo leſs than the triangle OFL; and the area of any circumſcribed polygon is always greater than that of the circle, and alſo greater than the triangle OFL: And becauſe theſe polygons may approach ſo near to equality as no difference can be aſſigned, it is evident, that the circle and the triangle OFL, which are always greater than the one, and leſs than the other, muſt be equal.

COR. I.

174. Hence, the area of a circle is equal to half the rectangle made of the circumference, and the radius, or to one fourth part of the rectangle made of the circumference and the diameter.

COR. II.

175. Since the area of the whole circle is equal to half the rectangle made of the circumference, and the radius, it is manifeſt, that the area of any ſector is equal to half the rectangle, made of the radius and the arc which terminates the ſector.

As theſe two laſt theorems have been immedi-

ately demonſtrated from the definition of proportions, we preſume that it is more agreeable to the ſtrictneſs of geometrical reaſonings, than any other method that has hitherto appeared.

SECT. VIII.

Of *SOLIDS.*

DEFINITIONS.

1. A Sphere is a ſolid deſcribed by the rotation of a ſemi-circle about its diameter as an axis.

2. A Cube is a ſolid contained between ſix equal ſquares.

3. A Parallelepipedon, is a ſolid contained between ſix parallelograms, the oppoſite ones being parallel and equal.

4. A Priſm is a ſolid whoſe oppoſite baſes are two equal and parallel polygons, and the ſides parallelograms.

5. A Cylinder is a ſolid whoſe oppoſite baſes are equal and parallel circles, and the ſection thro' the centers of the oppoſite baſes is always a parallelogram.

6. A Pyramid is a ſolid whoſe baſe is any polygon, and terminates in a point, whoſe ſides are triangles.

7. A Cone is a ſolid whoſe baſe is a circle, terminating in a point in the ſame manner as a pyramid; and any ſection through the upper point and the center of the baſe is always a triangle.

8. A Fruſtrum of a pyramid or cone is the remainder

mainder of a greater from which a leſſer has been cut off by a plane parallel to the baſe.

9. The upper point of a pyramid or cone is called the Vertex.

10. The line which joins the centers of the oppoſite baſes, in a priſm, cylinder, or the vertex to the center of the baſe, in a cone or pyramid, is called the Axis.

11. From whence, ſolids are ſaid to be right or oblique, according as the axis is perpendicular or oblique to the baſe.

Therefore a right cylinder may be deſcribed by the rotation of a rectangle about one of its ſides as an axis; and a right cone by the rotation of a right angled triangle about one of its ſides as an axis.

12. Similar ſolids are thoſe which are comprehended by an equal number of ſimilar planes.

13. Similar cones or cylinders are thoſe whoſe axes incline equally on their baſes; and when the axes are proportional to the diameters of their baſes.

14. Similar and Equal ſolids are thoſe which are comprehended by an equal number of equal and ſimilar planes.

15. The Side of a cone or cylinder is the interſection of a plane paſſing through their axis, with their ſurface.

THEOREM I.

176. *Parallelepipedons* ACEM, ACGK, *which have the ſame or equal baſes*, ABCD, *and the ſame altitude; or, which is the ſame, being terminated by the ſame plane* MEHI, *parallel to the baſe* AC, *are equal.* Fig. 10.

Since CE, BH, are parallelograms by defin. the triangles BEG, CFH, are * equal, as well as

 their

* art. 135.

their oppoſite ones AMK, DLI; and becauſe EG is equal to FH, the parallelograms MEGK, LFHI, having alſo the ſame height, are likewiſe equal; as well as ABGK, DCHI, and ABEM, DCFL, as being the oppoſite ſides of the ſame parallelepipedon. Therefore the triangular priſms ABEGKM, DCFHIL, being terminated by the ſame number of equal and ſimilar planes, are equal. From which, taking the ſame priſm PFGKLQ, and adding the ſame ABCPQD to the remainders, we ſhall have the parallelepipedon ACEM equal to the parallelepipedon ACGK. Conſequently, parallelepipedons which have equal baſes and equal altitudes, are equal.

THEOREM II.

Fig. 11. 177. *If a parallelepipedon* HB *be cut by a plane* HDBF, *paſſing through the oppoſite angles, the two triangular priſms* EDF, GBF, *are equal.*

For the oppoſite planes AF, DG, and HA, GB, are parallel and equal by defin. the diagonals DB, HF, divide the parallelograms AC, EG, which are parallel and equal, * into equal and ſimilar triangles, and the plane DBFH being common to both, the priſms EDF, GBF, which are terminated by the ſame number of equal and ſimilar planes, are ſimilar and equal. Conſequently, the diagonal plane HDBF, divides the parallelepipedon into two equal and ſimilar triangular priſms.

* *art.* 31.

COR. I.

178. Hence, all triangular priſms, right or oblique, which have equal baſes and equal altitudes, are equal; ſince their doubles are * equal they muſt be ſo too.

* *art.* 54.

COR.

COR. II.

179. It is manifeſt, that all priſms HA whatſoever right or oblique which have equal baſes and equal altitudes, are equal; ſince they may be always divided into as many triangular priſms as their oppoſite baſes ABCDE, FGHIK, can be divided into triangles; and the parts in the one being equal to the reſpective parts in the other, their ſums, or the whole priſms will be equal. *Fig.* 12.

THEOREM III.

180. *If a priſm or cylinder be cut by a plane parallel to the baſe, the ſection will be ſimilar and equal to the baſe.*

Firſt, Let LMNOP be a ſection parallel to the baſe; then becauſe all the ſides of the priſm are prallelograms by defin. LM = KI, MN = IH, NO = HG, &c. ſince KL, IM, NH, are parallel and equal by ſuppoſition; but the angles L, M, N, O, are alſo equal to the reſpective angles K, I, H, G, as being formed by parallel lines in parallel planes; and as this will always happen let the plane paſs where it will; and figures, which have all their ſides and angles equal, are equal in all reſpects: It follows that the ſection LMNO, which is parallel to the baſe KIHG, is alſo equal to it. *Fig.* 12.

Secondly, Let ABCD be a ſection through the axis of the cylinder, AB, DC its ſides; draw the diameters AD, BC, and from any point E draw EF parallel to AD or BC; then becauſe ABCD, is always a parallelogram, by defin. EF will be equal to AD or BC: Since AE, DF, are parallel and equal by ſuppoſition, and as this always happens, wherever the line EF is drawn in any ſection of the cylinder, and the diameters *Fig.* 13.

meters of the bases being always the same, the section through FE, or through any other place, parallel to the base, will be always a circle, equal to the base.

THEOREM IV.

Plate VIII. *Fig.* 16. 181. *If a pyramid or cone be cut by a plane parallel to the base, the section will always be similar to the base.*

First, Let EFGH be a section parallel to the base ABCD; then by reason of the parallelism, we * have (KF:KB::)FE:BA::FG:BC, and (KG:KC::)GF:CB::GH:CD. Therefore, EF:BA::FG:BC::GH:CD, by equality * of ratios; and since the angles E, F, G, H, and A, B, C, D, are formed by parallel lines in parallel planes respectively, they are equal. Consequently, the planes EFGH, ABCD, having all their angles equal, and their sides proportional, are similar.

* *art.* 67.

* *art.* 53

Fig. 17. *Secondly*, If from any point C in the axis of the cone, a line CD be drawn parallel to the radius AB of the base drawn from the same side KB of the cone; then will * KC:KA::CD:AB; and since all the radii of the base are equal, all the lines drawn from the same point C, in the axis, in a plane parallel to the base, will also be always equal. Consequently, that plane, or the section parallel to the base, will always be a circle.

* *art.* 67.

THEOREM V.

Fig. 14. 182. *Parallelepipedons* AD, FI, *which have the same or equal bases* ABC, *are as their altitudes* AF, FG. *I say*, AF:FG::AD:FI.

In FG take FL equal to any parts $\frac{1}{4}$ AF of AE, and let a plane LMN pass through the point L, parallel to the base GHIK; then because parallel-

rallelepipedons, which have equal bases and equal altitudes * are equal, the parallelepipedon FN will be the same parts, $\frac{5}{4}$, of the parallelepipedon AD, as FL is of AF. Now it is evident, that if FL is less than FG, the parallelepipedon FN is less than FI; if FL is equal to FG, the parallelepipedon FN is equal to FI; and if FL is greater than FG, the parallelepipedon FN is also greater than FI: Since, therefore, the equal parts FL, FN, of the antecedents AF, AD, when compared to the consequents FG, FI, are both together less, equal to, or greater than the respective consequents; these quantities are proportional by defin. 5, of proportions.

* art. 176.

COR. I.

183. It is manifest that parallelepipedons AB, EL, which have equal altitudes DB, HL, are as their bases AD, EH; for with GD make the parallelogram CD equal to the base EH, compleat the parallelepipedon CB; then as the parallelepipedons CB, EL, have their bases and altitudes equal by supposition, * they are equal; and the parallelepipedons AB, CB, having the same base GB are as their altitudes AG, GC; or as * AD, CD; but the base CD has been made equal to the base EH; and the parallelepipedon EL is equal to CB. Consequently, AB : EL :: AD : EH.

Fig. 15.

* art. 176.

* art. 65.

COR. II.

184. Since every parallelepipedon is equal to two triangular * prisms whose bases are each half that of the parallelepipedon, and of the same altitude, whatever is true in the whole will also be true * in the halves. Consequently, triangular prisms, which have equal bases, are as their altitudes;

* art. 177.

* art. 54.

tudes; and if their altitudes are equal, they are as their baſes.

COR. III.

185. As any priſm whatſoever, right or oblique, may always be divided into as many triangular priſms as the baſe can be divided into triangles; and ſince whatever is true in reſpect to parts, is likewiſe true * in the wholes, it is evident that any two priſms, which have equal baſes, are as their altitudes; or, if their altitudes are equal, they are as their baſes; or, if any of their dimenſions are equal, they will always be as the unequal ones.

*art. 54.

LEMMA.

Fig. 18. 186. *If a plane* EH *moves parallel to itſelf, and any two parts* EF, GH, *of that plane, are conſtantly equal to each other in any ſituation whatſoever: I ſay, they will deſcribe equal ſolids at the ſame time.*

If it be denied that they are equal, they muſt be unequal in ſome parts; as the plane EH is parallel to the baſe AB, by ſuppoſition, the ſolids AEFB, AGHB, being terminated by parallel planes, have equal altitudes; they have alſo equal baſes AB; if they are unequal, it muſt be ſomewhere between the planes AB, EH; but the parts of a plane paſſing through theſe ſolids between AB, EH, parallel to the baſe AB, are always equal by ſuppoſition. Since, therefore, no parts of the ſolids AEFB, AGHB, can be unequal, they muſt neceſſarily be equal. Conſequently, ſolids deſcribed by equal parts of the ſame plane, moving parallel to itſelf, are equal.

COR

COR. I.

187. Hence, all cylinders, right, or oblique, which have the same or equal bases, and are terminated by the same plane, parallel to the base, are equal; for we have shewn * that any section parallel to the base is always a circle, equal to the base. Consequently, cylinders of equal bases and equal altitudes, are equal.

* art. 180.

The same thing is likewise true in respect to any prisms which have equal bases and equal altitudes.

COR. II.

188. It appears also, that a prism and a cylinder, which have equal bases and equal altitudes, are equal: For, since any sections in both, parallel to the base, are * always equal to the base, the bases being equal the sections will be equal; and since the altitudes are likewise equal by supposition, * the solids themselves will be equal. Consequently, a cylinder may be considered as a prism, and whatever is demonstrated in the one, will equally be so in the other.

* art. 180.

* art. 179.

COR. III.

189. Since prisms which have equal bases are as their altitudes, * or if their altitudes are equal, they are as their bases: Cylinders which have equal bases and altitudes, with prisms being equal to them, will likewise be as their altitudes, if their bases are equal, or as their bases, if their altitudes are equal.

* art. 185.

This property of prisms and cylinders may be deduced immediately from the definition of proportions, without having recourse to the article 185, since it entirely depends on the equality of solids which have equal bases and equal altitudes; but

but we have chosen to deduce them in this manner, in order to avoid the multiplicity of figures.

THEOREM VI.

190. *All cones or pyramids which have the same or equal bases, and equal altitudes, are equal.*

Fig. 18. Let the triangles ACB, ADB, represent the sections of two pyramids or cones, which have the same base AB and the same altitude, and let EH represent a section parallel to the base; then because of the parallels AB, EH, CD, we have
* *art.* 67. * DG : DA :: GH : AB, and DG : DA :: CE : CA :: EF : AB; therefore, GH : AB :: FF :
* *art.* 53. AB, by * equality of ratios: And hence, GH = EF. Now because EF and GH repre-
* *art.* 181. sent the diameters of * circles, in the cones, and two lines in the pyramids, similarly placed; and since these lines are constantly equal, the circles in the cone, or the similar planes in the pyramids, are likewise always equal. Consequently, the cones or pyramids, described by these similar and
* *art.* 186 equal planes, are * also equal.

COR.

191. It is manifest, that a cone is equal to a pyramid of an equal base and altitude; for since the ratio of the circle of any section in the cone, to the circle of the base, being equal to the ratio of the section of a plane in the pyramid to the base, which are equally distant from the base, and the bases being equal, these sections will be equal: And consequently, the solids described by these equal planes will be * equal; whence, whatever
* *art.* 186. is proved in respect to pyramids, will likewise be true in respect to cones.

THE-

THEOREM VII.

192. *A triangular prism* ABCDEF *is triple the pyramid* BFED *of the same base* FED *and the same altitude* BE.

Let two planes pass through FBC, and DBF, then because the diagonal CF divides the parallelogram DCAF into * two equal triangles CDF, CAF, the pyramids CAFB, CDFB, having equal bases CDF, and CAF, and the same vertex B, are * equal; and the pyramids CAFB, DEFB, having the bases CBA, DEF, which are equal by defin. and the same altitude EB, are * also equal. Therefore, since the pyramid CAFB is equal to both the pyramids CDFB, and DEFB, they are all three equal to each other. Consequently, the prism ABCDEF is triple the pyramid BFED, of the same base and altitude. *Fig.* 19. * *art.* 31. * *art.* 190. * *art.* 190.

COR. I.

193. Since any prism may be divided into triangular ones, and as a triangular pyramid is one third of the triangular prism of the same base and altitude; it follows that any pyramid is equal to one third of the prism of the same base and altitude.

COR. II.

194. As a cone is equal to a pyramid of an * equal base and altitude, and a cylinder equal to a prism of an * equal base and altitude, it is manifest that a cone is one third part of a cylinder of the same base and altitude. * *art.* 191. * *art.* 188.

COR. III.

195. Because prisms and cylinders, which have equal bases, are * as their altitudes; or, if their altitudes * *art.* 185.

altitudes are equal, they are as their bases; pyramids or cones, which have equal bases, are likewise as their altitudes; or, if their altitudes are equal as their bases, since the wholes are proportional to * their equal parts.

* *art.* 54

COR. IV.

196. Since the pyramid D B F E is one third part of the prism D C A B F E, it follows that the remainder D C B A F D will be the two thirds of that prism. This must be observed, because it serves hereafter to find the number of shot contained in a square or oblong Pile.

THEOREM VIII.

Fig. 15. 197. *If three lines* A, B, C, *are in a continued proportion, the rectangular parallelepipedon* E L, *made of these three lines, will be equal to the cube* F K, *made of the mean* B.

Let the base E H be equal to the rectangle of the extremes A, C, and the altitude H L equal to the mean B; then because ∴ A : B : C, the base E H is * equal to the base F I; and since the altitudes H L, I K, are also equal by supposition, the parallelepipedons E L, F K, are * equal. Consequently, if three lines are in a continued proportion, the rectangular parallelepipedon made of these three lines is equal to the cube of the mean.

* *art.* 73,

THEOREM IX.

198. *If four lines* A, B, C, D, *are in a continued proportion, the rectangular parallelepipedon* C B *made of the square of the first* A *and the fourth* D, *will be equal to the cube* F K *of the second* B.

Let C G be equal to the fourth D; G D, D B, equal to the first A; and let the base E H be equal

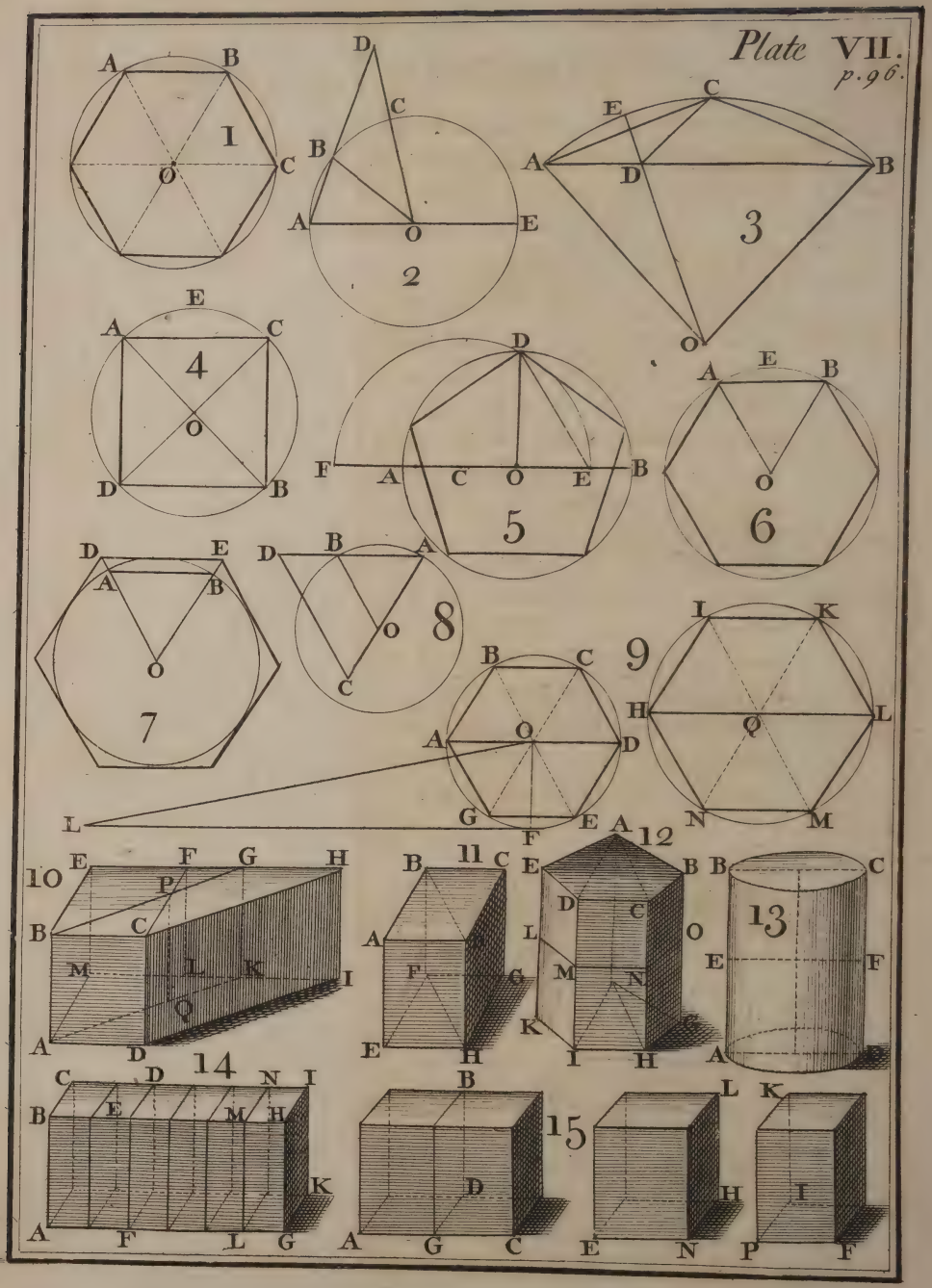
Plate VII.
p. 96.
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qual to the rectangle of the second B and the third C; and the altitude H L equal to the first A; then becauſe ∴ A : B : C : D, the baſe C D is * equal to the baſe E H; and ſince the altitudes D B, H L, are equal by conſtruction, the parallelepipedons C B, E L, which have their baſes and altitudes equal, are * equal; and as the parallelepipedon E L is made of three lines A, B, C, which are in a continued proportion, it will be equal * to the cube F K, made of the mean B. Since, therefore, the parallelepipedon E L is equal to both the cube F K and the parallelepipedon C B, the cube F K will be equal to the parallelepipedon C B. Conſequently, if four lines are in a continued proportion, the parallelepipedon C B, made by the ſquare of the firſt and the laſt, is equal to the cube F K of the ſecond.

* *art.* 72.
* *art.* 176.
* *art.* 197.

THEOREM X.

199. *If four lines* A, B, C, D, *are in a continued proportion, the cube* A B, *of the firſt* A, *will be to the cube* F K *of the ſecond* B, *as the firſt line* A *is to the fourth* D.

Let G C be equal to the fourth line D, and compleat the rectangular parallelepipedon C B, on the baſe G B; then becauſe the parallelepipedon C B is made of the ſquare G B of the firſt, and its altitude G C is equal to the laſt, it will be equal to the cube * F K made of the ſecond B; but A B, and C B, having the ſame baſe G B, are as their * altitudes: Therefore A B : C B, or F K :: A G : G C, or as A : D. Conſequently, if four lines are in a continued proportion, the cube of the firſt is to the cube of the ſecond, as the firſt line is to the fourth.

* *art.* 198.
* *art.* 182.

COR.

200. Hence, cubes are in a triplicate proportion to that of their ſides; ſince, if four lines are in a continued proportion, the ratio of the firſt to the laſt is ſaid to be triplicate to that * of the firſt and ſecond.

*def. 8, *of proportion.*

THEOREM XI.

201. *If there are ſix lines* A, B, C, D, E, F, *proportional, the cube* AB *of any antecedent* A, *is to the cube* FK, *of its conſequent* B, *as the rectangular parallelepipedon* CB, *of the antecedents* A, C, E, *is to the rectangular parallelepipedon* EL *of the conſequents* B, D, F.

Let the baſe CD be equal to the rectangle of the ſecond and third antecedents C, E, and the baſe EH equal to the rectangle of the ſecond and third conſequents D, F; compleat the parallelepipedons CB, EL, with the ſame altitudes of the cubes AB, FK; then as the ſolids AB, CB, have the ſame altitudes, they are * as their baſes AD, CD; and the ſolids FK, EL, having the ſame altitudes, are alſo as their baſes FI, EH.

* *art.* 182.

But when ſix lines are proportional, the ſquare AD of the firſt is to the ſquare FI of the ſecond, * as the rectangle CD of the ſecond and third antecedents is to the rectangle EH of the ſecond and third conſequents: Therefore AB : FK :: CB : EL, ſince theſe ſolids have been proved to be proportional to their baſes. Conſequently, if ſix lines are proportional, the cube AB of any antecedent A, is to the cube FK of its conſequent B, as the parallelepipedon CB, of the antecedents, is to the parallelepipedon EL of the conſequents.

* *art.* 81.

THE-

THEOREM XII.

202. *Rectangular parallelepipedons* AB, FK, *which have their bases and altitudes reciproque are equal*; *if* PK : GB :: AG : PF. *I say*, AB = FK.

In AG produced, take GC equal to PF; compleat the solid CB, upon the base GB, and altitude CG; then because the solids FK, CB, have equal altitudes GC, PF, by construction, they are as their bases; that is, FK : CB :: PK : GB; and the solids AB, CB, having the same base GB, are as their altitudes; *viz.* AB : CB :: AG : GC; but PK : GB :: AG : PF, by supposition; and as GC, PF, are equal by construction, we have FK : CB :: AB : CB, by equality of * ratios, whence * FK = AB. Consequently, rectangular parallelepipedons, which have their bases and altitudes reciproque, are equal.

* *art.* 53.
* *art.* 52.

COR. I.

203. Since all parallelepipedons, right or oblique, which have equal bases and equal altitudes, are equal; it follows that all parallelepipedons, right or oblique, which have their bases and altitudes reciproque, are also equal.

COR. II.

204. Since a parallelepipedon is equal to two triangular prisms, and a cylinder or prism of any base equal to as many triangular prisms as the base may be divided into triangles; it is evident, that all prisms or cylinders, which have their bases and altitudes reciproque, are equal.

COR. III.

205. Because a pyramid or cone is one third of the prism or cylinder of the same base and altitude; and since whatever is true in respect to the

*art. 54. whole's, * is likewise so in respect to the parts, it follows, that pyramids or cones, which have their bases and altitudes reciproque, are also equal.

THEOREM XIII.

206. *Similar parallelepipedons* EL, FK, *are as the cubes of their homologous sides* EN, PF.

For the cube of EN, and the solid EL, having the same altitude EN, are to each other as the square of EN to the rectangle NL; and the cube of PF and the solid FK, having the same altitude PF, are to each other as the square of PF to the rectangle PK: But by the similarity of the solids EL, FK, we have EN : PF :: NH : PI, *art. 87. and EN : PF :: HL : IK. Therefore, * the square of EN, is to the square of PF, as the rectangle NL of the remaining antecedents is to the rectangle PK of the remaining consequents; and since the solids are proportional to these planes, the cube of EN will be to the solid EL, as the cube of PE is to the solid FK. Consequently, similar parallelepipedons are to each other as the cubes of their homologous sides.

COR. I.

207. It is evident, that similar triangular prisms are to each other as the cubes of their homologous sides; since they are the halves of parallelepipe- * art. 54. dons, of the same altitudes, and a base double the base of the prism, and the parts are * proportional to their wholes.

COR. II.

208. Hence, all similar prisms and cylinders are as the cubes of their axes or homologous sides; since the prisms may always be divided into tri- *art. 188. angular ones; and the cylinders being equal to * prisms of equal bases and equal altitudes; the whole's

whole's being in the ſame ratio as any of their * like parts. * art. 54.

COR. III.

209. Hence alſo, all ſimilar pyramids or cones are as the cubes of their axes, ſince they are one third of the priſms or cylinders of the ſame baſes and altitudes, the like parts being in the ſame ratio as their wholes.

COR. IV.

210. It is manifeſt, that all ſimilar ſolids are in the triplicate ratio to that of their homologous ſides; for ſince they are as the cubes of theſe ſides, and the cubes being in that ratio, * the ſolids themſelves will be in the ſame ratio, * art. 200.

THEOREM XIV.

211. *The fruſtrum* ALKB, *whoſe baſe* AC *is a ſquare, is equal to a pyramid of the ſame baſe and altitude, and to another of the ſame altitude, and whoſe baſe is a rectangle made by the ſum* AB, GH, *of the ſides of the oppoſite baſes, and the ſide of the leſſer baſe.* Fig. 20.

It is evident, that this fruſtrum is equal to a pyramid of the ſame baſe AC, and the ſame altitude EF, and to four pyramids ſuch as ABHEG, whoſe baſe is the ſide AH, and whoſe vertex the point E.

Now becauſe the baſe AH is equal to a rectangle made of the ſum of the parallel ſides AB, GH, * and of the perpendicular drawn in that plane to theſe parallel ſides. * art. 51.

Let FESM be a ſection through the axis FE, perpendicular to the ſide AH; ſo that EF repreſents the axis, SM the perpendicular to the parallel ſides, and ER perpendicular to MS produced,

the altitude or perpendicular drawn from the vertex E, perpendicular to the base ABHG produced: If AB + GH = A, then will * $\frac{A \times SM \times ER}{2 \times 3}$, * *art.* 193.
expreſs the pyramid AGEHA; and if SN, EP, are drawn parallel to EF, and SM, the rectangle SF will be equal to the parallelogram EM, or $ES \times EF = ER \times SM$: Therefore $\frac{A \times SM \times RE}{2 \times 3}$ will be equal to $\frac{A \times EF \times ES}{2 \times 3}$; this will expreſs one of theſe pyramids: And ſince ES is equal to the perpendicular drawn from the point E to GH, it will be equal to half of the ſide GL or GH; if therefore, we ſuſtitute $4ES$, or $2GH$, for ES, we ſhall have $\frac{A \times GH \times EF}{3}$ for the expreſſion of the four pyramids, whoſe baſes are in the ſides of the fruſtrum, and whoſe vertex is in the point E. Conſequently, the fruſtrum is equal to a pyramid of the ſame baſe ABCD, and altitude FE; and to another, whoſe baſe is a rectangle made of the ſum of the ſides AB, GH, and the ſide GH; and whoſe altitude is EF, that of the fruſtrum.

COR. I.

212, Hence it is evident, that the ſquare fruſtrum AK, is equal to a pyramid whoſe baſe is $\overline{AB}^2 + AB \times GH + \overline{GH}^2$, and whoſe altitude is the ſame as that FE of the fruſtrum: Whence the baſe is equal to the ſum of the baſes of the fruſtrum, and to a mean proportional between theſe planes.

COR

COR II.

213. Since all fruſtrums of cones or pyramids of any baſe, are equal to a ſquare fruſtrum, whoſe oppoſite baſes and altitudes are equal, it follows that the fruſtrum of any pyramid or cone is equal to a pyramid whoſe baſe is equal to the ſum of the oppoſite baſes of the fruſtrum and a plane which is a mean between theſe planes, and whoſe altitude is equal to that of the fruſtrum.

N. B. The mean plane in the fruſtrum of a cone, is equal to half the rectangle made by the radius of one baſe and the circumference of the other; and in a fruſtrum whoſe oppoſite baſes are ſimilar polygons, to half the rectangle made by the perpendicular drawn from the center in one to one of the ſides, and the ſum of the ſides in the other.

For if a, b, are the radii in the cone, or the perpendiculars in the pyramid c, d, the circumferences or ſums of their reſpective ſides: Then will ac, bd, be double the upper and * lower baſes, and bc double the mean plane: For we have * $a:b::c:d$, by ſuppoſition, and * $ac:bc::a:b$, as likewiſe $cb:bd::c:d$; therefore, * $ac:bc::bc:bd$, by * equality of ratios.

* *art.* 174.
* *art.* 171.
* *art.* 54.
* *art.* 53.

THEOREM XV.

214. *If about the quadrant* AMB *of a circle, a ſquare* ADBO *be deſcribed, and any line* PN, *be drawn parallel to the radius* OB, *meeting the diagonal* OD *in* L, *the circumference in* M, *and* BD *in* N; *the ſegment of the ſphere deſcribed by the ſegment* PMBO *in the rotation of the figure about the axis* AO, *will be equal to the ſolid deſcribed by the ſpace* LNBO *in that rotation.* Fig. 21.

Becauſe AD is equal to AO, PL will be equal to OP; and by reaſon of the right angled triangle OPM, the difference between the ſquares of OM and OP, or PL, will be equal to the ſquare of PM; or becauſe OM is equal to PN by the property of the circle, the difference between the ſquares of PN, and PL, will be equal to the ſquare of PM.

* *art.* 170. Now becauſe circles are as the ſquares of their *radii, the difference between the circles deſcribed by PN and PL in the rotation, or the area deſcribed by LN, will be equal to the circle deſcribed by PM; and as this property is general, the ſolid deſcribed * *art.* 186. by LNBO is equal to * the ſolid deſcribed by PMBO.

COR. I.

215. Hence any ſegment PMBO of a ſphere is equal to the difference between the correſponding cylinder deſcribed by the rectangle PNBO, and the cone deſcribed by the triangle OPL. Conſequently, the ſemi-ſphere deſcribed by the quadrant AMBO, is equal to the difference between the circumſcribed cylinder deſcribed by the rectangle ADBO, and the inſcribed cone, deſcribed by the triangle OAD.

COR. II.

216. Since the cone deſcribed by the triangle * *art.* 194. OAD is * one third of the cylinder deſcribed by the rectangle ADBO, it is manifeſt, that the ſemi-ſphere is the two thirds of that cylinder. Conſequently, the circumſcribed cylinder, the ſemi-ſphere, and the inſcribed cone, are to each other as the numbers 3, 2, 1.

COR.

COR. III.

217. As the cones described by the triangles OPL, OPM, OPN, in the rotation of the figure about the axis AO, have the same altitude PO, they * are as their bases, or as the squares of the radii * PL, PM, PN; and since the square of PN is equal to the sum of the squares of PL, PM; the cone described by OPN will be equal to the sum of the cones described by OPL, OPM. But the sum of the segment of the sphere PMBO and the cone OPL, is * equal to the cylinder OBNP; if therefore the cone OPN be taken from the cylinder OBNP, and the two cones OPL, OPM, from its equal the segment of the sphere and the cone OPL; the solid described by OBN, will be equal to the solid described by the sector OBM.

* art. 195.
* art. 170.
* art. 214.

COR. IV.

218. Hence, if these equal solids are supposed to be made up by an infinite number of small pyramids, their bases being in the surfaces described by BN, and BM, and vertexes in the center O: then the sum of all the pyramids in the one being equal to the sum of all those in the other; and their altitude OB being the same; the sum of the bases in the one will be equal to the sum of the bases in the other. Consequently, the surface of any segment OBMP, of a sphere is always equal to the surface of the corresponding part PNBO, of the circumscribed cylinder, and the surface of the whole sphere equal the convex surface of the cylinder.

COR. V.

219. It is manifest, that all spheres are similar, and therefore as the cubes of * their axis or diameters; or in the triplicate * proportion;

* art. 208.
* art. 210.

tion to that of their diameters; since the diame-
* *art.* 171. ters of circles are as * their circumferences.

THEOREM XVI.

Fig. 24. 220. *The convex surface of a right cylinder is equal to a rectangle made of the circumference of the base and its axis.*

For if we suppose this surface to unfold itself into a plane one, it is evident, that it would make a rectangle, whose base being equal to the circumference of the base of the cylinder and altitude equal to one of the sides or the axis; since all the sides are equal to each other and to the axis, by the definition of a right cylinder.

COR. I.

221. Hence, as the surface of the sphere is equal to the convex surface of the circumscribed cylinder, and the surface of that cylinder equal to the rectangle of the circumference of a great circle of the sphere and the diameter of the sphere; it follows, that the surface of a sphere is equal to a rectangle of the circumference of a great circle and its diameter.

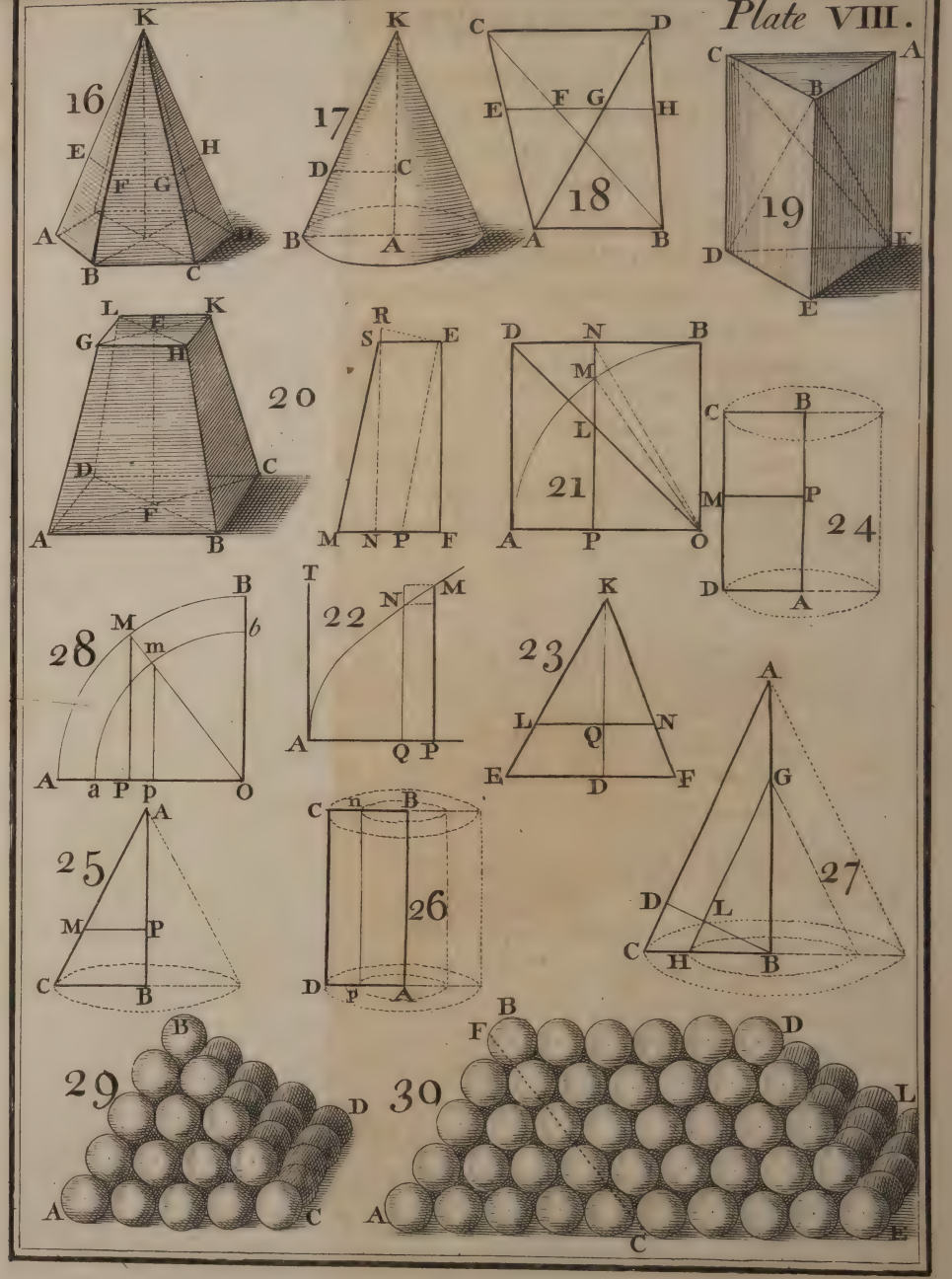
COR. II.

* *art.* 174. 222. The area of a circle being * the fourth part of a rectangle whose base is the circumference, and altitude the diameter, it is evident, that the surface of the sphere is quadruple the surface of its great circle.

COR. III.

* *art.* 170. 223. Since the circles are as * the squares of their radii, and the square of a line double another line, is quadruple, the square of half that line; it is manifest, that the surface of a sphere is also equal

Plate VIII.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qual to a circle, deſcribed with the diameter of the ſphere as a radius.

COR. IV.

224. Hence, the ſurfaces of ſimilar cylinders, and of ſpheres, are to each other as the ſquares of their axes or diameters, or in the duplicate * ratio * to that of their axes: For the axes being proportional in the cylinders to the diameters of the baſes, and the diameters of circles as their circumferences, they are as the ſquares * of their homologous ſides * or axes. *art.* 77. *art.* 79.

THEOREM XVII.

225. *The convex ſurface of a right cone is equal to a circular ſector, whoſe radius is the ſide of the cone, and the arc equal to the circumference of the baſe of the cone.* Fig. 17.

For if this ſurface be ſuppoſed to unfold itſelf into a plane one, it is evident, that, as all the ſides BK of the cone are equal, it would form a circular ſector, whoſe radius is equal to the ſide BK, and the arc equal to the circumference of the baſe of the cone.

COR.

226. Hence, the ſurfaces of right ſimilar cones are as the ſquares of their axes; for as they are equal to ſimilar ſectors, which are as the ſquares of the radii or ſides BK, and theſe ſides are as the axes AK, in ſimilar cones.

SECT.

SECT. IX.

Of the INVESTIGATION of *AREAS* and *SOLIDS*.

AS we have ſhewn in the former ſections the Manner of comparing ſurfaces and ſolids, and the proportions which they bear to each other, it may ſeem ſuperfluous to treat the ſame ſubject over again: But as the former diſquiſitions are limitted, and extend no farther than to certain obvious figures, we think it will not altogether appear uſeleſs to the curious reader, to give here a more general method, which extends to all kinds of figures whoſe properties are given; beſides, the principles laid down here in this ſection are the foundation of the moſt compendious and extenſive method of fluxions and fluents, invented by Sir *Iſaac Newton*.

PROB. I.

Fig. 22. 227. *If the area* ANQ, *terminated by the perpendicular* NQ *to the baſe* AQ, *be expreſſed by the baſe* AQ *and conſtant quantities, in ſuch a manner, that when the baſe* AQ *becomes* AP, *the area* ANQ *becomes* AMP; *it is required to find the perpendicular* PM *from the given area* AMP.

Subtract the area ANQ from AMP, and divide the difference QNMP by the difference QP of the baſes; then the quotient will give a line greater than QN and leſs than PM: For the rectangle of that quotient and QP, is equal to the area

area QNMP, and the rectangle PN is less, and QM greater than that area; and as they have the same base QP, the altitude QN will also be less, and PM greater than the said quotient.

Now as this always happens when QN becomes equal to PM, the quotient which is always greater than the one and less than the other, will likewise become equal to the perpendicular PM required.

PROB. II.

228. *If the figure* AM *revolves about the axis* AP, *to find the plane or circle described by* PM, *in that rotation, from the given solid described by the area* AMP.

From the solid described by AMP, subtract the solid described by ANQ, and divide the difference described by QNMP, by the difference QP of the bases; then the quotient will, by the same argument as before, be a plane or circle greater than that described by QN, and less than that described by PM; and when AQ becomes AP, and QN equal to PM, this quotient will become equal to the circle required, described by PM.

It may be observed that this problem not only extends to any solids described by the rotation of the figure AMP about the axis AP, but likewise to all solids whose sides of similar sections are always expressed by QN or PM; for the demonstration of this case is exactly the same with the former.

PROB. III.

229. *If the figure* AMP *revolves about the line* AT *perpendicular to the base* AP; *to find the surface described by* PM, *from the given solid described by the area* AMP.

From

From the ſolid deſcribed by AMP, ſubtract the ſolid deſcribed by ANQ, and divide the difference deſcribed by ONMP, by the difference QP, of the baſes; then by the ſame argument as in the firſt problem, the quotient will give a ſurface greater than that deſcribed by QN, and leſs than that deſcribed by PM; and when the baſes AQ, AP, or the perpendiculars QN, and PM, become equal, the quotient will become the ſurface required, deſcribed by PM.

COR. I.

230. Hence, if $AQ=z$, $AP=x$; and a, b, c, d, expreſs conſtant quanties, then if $az+bzz+cz^3+dz^4$, expreſſes the area ANQ, or the ſolid deſcribed by that area; $ax+bxx++cx^3+dx^4$, will, by the condition of the problem, alſo expreſs the area AMP, or the ſolid deſcribed by that area.

Now if we ſubtract the former from the latter, we ſhall have $ax-az+bxx-bzz+cx^3-cz^3+dx^4-dz^4$; which being divided by QP, or $x-z$, gives $\frac{ax-az}{x-z}=a$.

$$\frac{bxx-bzz}{x-z}=bx+bz.$$

$$\frac{cx^3-cz^3}{x-z}=cxx+cxz+czz.$$

$$\frac{dx^4-dz^4}{x-z}=dx^3+dxxz+dxzz+dz^3.$$

Therefore $a+bx+bz+cxx+cxz+czz+dx^3+dxxz+dxzz+dz^3$, will be the quotient; and when AQ becomes equal to AP, or $x=z$,

$x=z$, then this expreſſion gives $a+2bx+3cxx+4dx^3$, for the perpendicular or plane required.

COR. II.

231. Since the area or ſolid $ax+bxx+cx^3+dx^4$, gives $a+2bx+3cxx+4dx^3$, for the perpendicular or plane; that is, ax gives a, bxx gives $2bx$, cx^3 gives $3cxx$, and dx^4 gives $4dx^3$. It is evident, that if the area or ſolid AMP, be expreſſed by any whole powers, of the baſe AP and conſtant quantities, the perpendicular or plane will be found by this

GENERAL RULE I.

232. *Multiply every term by the exponent of the powers of the baſe, and divide the whole by the baſe.*

On the contrary, the perpendicular or plane PM, being expreſſed by any whole powers of the baſe AP, and conſtant quantities, the area or ſolid AMP will be found by this.

GENERAL RULE II.

233. *Multiply every term by the baſe, and divide each term by the exponent of the baſe thus increaſed.*

Thoſe who are deſirous to ſee a general demonſtration of the rule, that x^n, gives nx^{n-1}, for the perpendicular may find it in my treatiſe of conic-ſections, pa. 72, art. 162 and 163.

As nothing illucidates more general rules than their application to eaſy and familiar examples, we ſhall give ſome of the plaineſt firſt, and then proceed gradually to thoſe for which this Inveſtigation was intended.

EX-

EXAMPLE I.

Fig. 23. 234. *To find the area of a triangle* EKF.

Let KD be perpendicular to the base EF, draw LN any where parallel to EF, intersecting DK in Q; then if the constant quantities EF $=a$, KD $=b$, and the variable KQ $=x$, the parallel-

*art. 67. ism gives * KD(b) : EF(a) :: KQ(x) : LN $= \frac{ax}{b}$.

Now the perpendicular LN to KD being $\frac{ax}{b}$,

*art. 233 we have $\frac{axx}{2b}$ * by the second general rule for the area of the triangle KLN; and when KQ becomes KD, or $x=b$, $\frac{axx}{2b}$ will give $\frac{ab}{2}$ for the area of the triangle EKF, which therefore is equal to half the rectangle of the same base and altitude.

The similar triangles KLN, KEF, are to each other as $\frac{axx}{2b}$ to $\frac{ab}{2}$, which is as xx to bb; that is, as the squares of their homologous sides.

EXAMPLE II.

Fig. 24. 235. *To find the solid content of a right cylinder described by the rotation of a rectangle* ABCD *about one of its side* AB *as an axis.*

From any point P in the axis, draw PM parallel to AD; then because AD, BC, are parallel and equal, PM is also equal to AD; therefore the circles described by the equal radii AD, PM, in the rotation of the figure are also equal: If a expresses the circle of the base, and AP $=x$,

*art. 233. then * $\frac{ax}{1}$, or only ax will express the cylinder de-

described by APMD, and $a \times AB$ the cylinder described by ABCD.

COR. I.

236. If an oblique cylinder be described by the parallel motion of a circle, so as the center describes a right line, it is evident that its solid content will be found in the same manner as before. Consequently, if their bases and altitudes are equal, the cylinders are equal.

COR. II.

237. If a prism is supposed to be described by the parallel motion of a polygon, so as its center describes a right line, it is manifest that its solid content will be found in the same manner as that of a cylinder. Therefore prisms whose bases and altitudes are equal, are equal.

COR. III.

238. If a, b, are the radii of the bases in cylinders, or the perpendiculars, drawn from the centers of the polygons to one of their sides in prisms, c, d, the circumferences of the bases, or number of sides, m, n, the altitudes; then will $\frac{ab}{2}$, $\frac{bd}{2}$, express * the areas of the bases, and $\frac{acm}{2}$, $\frac{bdn}{2}$, the cylinders or prisms; when the altitudes m, n, are equal, they will be as their bases $\frac{ac}{2}$, $\frac{bd}{2}$, or when the bases $\frac{ac}{2}$, $\frac{bd}{2}$, are equal as their altitudes m, n; or lastly, if they are similar they will be as the cubes of their altitudes, m, n; for since $m : n :: a : b$, and $m : n :: c : d$, by supposition, then * $mm : nn :: ac : bd$; but $m^3 : acm :: mm : ac$, * and $n^3 :$

* art. 174.

* art. 80.

* art. 189.

$n^3 : bdn :: nn : bd$: Therefore $m^3 : acm :: n^3 : \frac{acm:}{2}$ bdn, by equality * of ratios; or * $m^3 : n^3 :: \frac{acm:}{2} : \frac{bdn.}{2}$

* art. 53.
* art. 57.

EXAMPLE III.

Fig. 25. 239. *To find the solid content of a right cone described by the rotation of a right angled triangle* ABC, *about one of its sides* AB *as an axis.*

From any point P in the axis, draw PM parallel to BC, and let the constant quantities $AB = b$, $BC = a$; the circumference of the radius BC, $= c$, and the variable quantity $AP = x$; then will * $AB(b) : BC(a) :: AP(x) : PM = \frac{ax}{b}$;

* art. 67.

and because the circumferences are * as their radii, we have $a : c :: \frac{ax}{b} : \frac{cx}{b}$ = to the circumference described by PM; and $\frac{ax}{2b} \times \frac{cx}{b}$ or $\frac{acxx}{2bb}$ will * be the area of the circle described by PM. Therefore * $\frac{acx^3}{6bb}$ expresses the solid content of the cone described by APM; and when AP becomes AB, or $x = b$; then will $\frac{acx^3}{6bb}$ give $\frac{abc}{6}$ for the cone described by ABC.

* art. 171.

* art. 174.

* art. 233.

COR. I.

240. Hence the similar cones described by APM, ABC, are to each other as $\frac{acx^3}{6bb}$ to $\frac{abc}{6}$; that is, as * x^3 to b^3, which shews that similar cones are as the cubes of their axes.

* art. 54.

COR.

COR. II.

241. As the cylinder of the same base and altitude is expressed by * $\frac{abc}{2}$, the cone $\frac{abc}{6}$ will be one third part of the cylinder of the same base and altitude. * art. 235.

COR. III.

242. If the cone was oblique, so that ABC represents a section through the axis, then MN parallel to AC, will represent a diameter of a circle; and therefore the cone would in the same manner as above be found to be one third part of the cylinder of the same base and altitude. Consequently all cones, right or oblique, which have equal bases and equal altitudes, are equal: If cones have equal bases, they are as their altitudes; * and if their altitudes are equal, they are as their bases. Fig. 23. * art. 54.

The same thing may be said in respect to pyramids.

EXAMPLE IV.

243 *To find the surface of a right cylinder.*

From any point p in the base AD draw pn parallel to the axis AB, then if we suppose Ap to be the base of the cylinder described by the rectangle ABnp, about the axis AB, as in the third problem, we are to find the surface described by pn, from the given solid Ap. If $AD = a$, $AB = b$, the circumference of the radius $AD = c$, and if $An = x$, then because *AD (a) : Ap (x) :: $c : \frac{cx}{a}$ = to the circumference of the radius Ap; Fig. 16. * art. 171.

* $\frac{cxx}{2a}$ will be the area of the circle described * art. 174.

*art. 188. by Ap; and hence, $\frac{bcxx}{2a}$ will * expreſs the cylinder deſcribed by An; and therefore by the * art. 232. * firſt general rule $\frac{bcx}{a}$ will expreſs the ſurface deſcribed by pn, and when Ap becomes AD, or $a=x$, then $\frac{bcx}{a}$ gives bc for the ſurface deſcribed by DC. Which conſequently is equal to a rectangle of the circumference of the baſe and the axis of the cylinder.

EXAMPLE V.

Fig. 27. 244. *To find the ſurface of a right cone deſcribed by* AC.

Draw any where HG parallel to CA, and BD perpendicular to CA, interſecting HG in L; then if BL be taken as the baſe, we are to find the ſurface deſcribed by HG, from the given ſolid BHG, according to prob. the third.

If $BD=d$, $BL=x$, $AC=g$, $AB=a$, $BC=b$, and c the circumference of the radius BC; then the right angled ſimilar triangles ABC, * art. 69. BLG, give * $BC(a):AC(g)::BL(x):BG=\frac{gx}{a}$; and by reaſon of the parallels AC, * art. 67. GH, we have * $BD(d):BC(a)::BL(x):BH=\frac{ax}{d}$; and ſince the circumferences are as * art. 171. their radii * $BC(a):BH\frac{(ax)}{d}::c:\frac{cx}{d}=$ to the circumference of the radius BH, and $\frac{ax}{2d}\times\frac{cx}{d}$ or $\frac{acxx}{2dd}$ will be the area of the circle deſcribed by

by BH, which being multiplied by one third of * BG $\frac{(gx)}{a}$ gives $\frac{gcxxx}{6dd}$ for the solid described by BHG; therefore $\frac{3gcxx}{6dd}$ or $\frac{gcxx}{2dd}$ will be the surface described by HG, by rule * the first; and when BL becomes BD, or $x=d$, then $\frac{gcxx}{2dd}$ will give $\frac{gc}{2}$ for the surface described by AC. Which consequently is equal to half the rectangle made by the circumference of the base and the side AC of the cone.

*art. 239.

*art. 232.

COR.

245. Hence, if the base EF is equal to the circumference of the cone's base, KD to the side of the cone; then the triangle EKF will be equal to the surface of the cone; and the triangle KLN equal to any part described by AM, if KQ = AM: And therefore the remainder ELNF of the triangle, to the remainder of the surface described by MC of the cone. But the area ELNF, is equal to the rectangle of half the * sum of the parallel sides EF, LN, and the perpendicular DQ; or to the rectangle of an arithmetical mean between EF, LN, and DQ; and as DQ is = MC, by supposition, EF, to the circumference described by the point C, and LN to the circumference described by M. Therefore the rectangle made by MC, and the circumference described by the point which bisects that line, will be equal to the surface described by MC.

Fig. 23.

Fig. 25.

* art. 51.

EXAMPLE VI.

Fig. 28. 246. *To find the solid content of a segment of a sphere described by the segment of a circle* AMP, *about the radius* AO *as an axis.*

Let the radius $AO = a$, its circumference c; $AP = x$, $PM = y$; then by the property of the circle * we have $2a - x : y :: y : x$, or $yy = 2ax - xx$; and because the radii are as their circumferences, * $AO(a) : PM(y) :: c : \frac{cy}{a}$ = to the circumference of the radius PM, and $\frac{cyy}{2a}$ or its equal $\frac{c}{2a} \times \sqrt{2ax - xx}$, equal to the area described by PM; therefore * $\frac{c}{2a} \times \sqrt{2ax - \frac{1}{3}x^3}$ by rule the second will express the solid described by AMP; and when AP becomes equal to AO, or $a = x$, then that expression will give $\frac{c}{2a} \times \frac{2a^3}{3}$, or $\frac{aac}{3}$ for the solid content of the semi-sphere: This is the same expression as has been found in art. 216.

* art. 117. * art. 73. * art. 171. * art. 233.

COR.

Fig. 21. * art. 217. 247. Since the solid described by OBN is * equal to the solid described by the sector OBM, and the solid described by OBD, equal to the semi-sphere; therefore the solid described by OND equal to the solid described by OMA: But the solid described by OND is equal to the difference between the solids described by ONDA and ODA; and as the solid by ONDA is to the cone described by ODA, as $AP + \frac{1}{3}PO$, to $\frac{1}{3}AO$; and therefore their difference as $\frac{1}{3}AP$;

$\frac{2}{3}$ AP; that is, the solid described by OND, or by the sector OMA, is equal to $\frac{acx}{3}$.

EXAMPLE VII.

248. *To find the surface of a sphere.* *Fig.* 28.

From the center O, with any radius Oa, describe another arc a m *b*, intersecting the radius OM in m, draw mp perpendicular to AO; then if Oa is taken for the base of the solid described by Oma, about OA, we are to find the surface described by am, according to prob. the third; for ma is always perpendicular to the base Oa, and parallel to MA.

If Om, or Oa = *z*, the similar triangles OPM, Opm, give * OM (*a*) : OP (*a*—*x*) : : Om (*z*) : Op = $\frac{az - xz}{a}$ or pa = $\frac{zx}{a}$; and because the circumferences are as their radii,* OA (*a*) : Oa (*z*) : : *c* : $\frac{cz}{a}$ = to the circumference of the radius Oa. Now these values being substituted into $\frac{acx}{3}$; viz. *z*, $\frac{cz}{a}$, and $\frac{zx}{a}$, for *a*, *c* and *x*, we shall have $\frac{cxz^3}{3aa}$, for the solid described by the sector Oma: Therefore by rule the * first $\frac{3cxzz}{3aa}$, or $\frac{cxzz}{aa}$, will be the surface described by the arc am; and when Oa becomes OA, or *z* = *a*, then $\frac{cxzz}{aa}$ will give $\frac{cxaa}{aa}$, or *cx*, for the surface described by the arc AM, and *ac* for that of the semi-sphere, which is the same as in art. 218.

* art 69.

* art. 171.

* art. 232.

LEMMA.

249. *To find the square root of* $1-xx$, *or, which is the same, to find an infinite series equal to* $\sqrt{1-xx}$.

Suppose $\sqrt{1-xx}=1+ax^2+bx^4+cx^6+dx^8+\&c.$

$1+ax^2+bx^4+cx^6+dx^8+\&c.$

This squared gives $1+ax^2+bx^4+cx^6+dx^8$

$ax^2+aax^4+abx^6+acx^8$

$bx^4+abx^6+bbx^8$

cx^6+acx^8

$dx^8.$

$1-xx=1+2ax^2+\overline{aa+2b}x^4+\overline{2ab+2c}x^6+\overline{bb+2ac+2d}x^8+\&c.$

Now if the coefficients of the first and second terms on each side are supposed equal, and each of the rest equal to o, it is plain that the equality will be preserved, since it gives $1-xx=1-xx$.

$$\left.\begin{aligned} -2a&=1.\\ 2b+aa&=o\\ 2ab+2c&=o\\ 2d+2ac+bb&=o \end{aligned}\right\} \text{ and } \left\{\begin{aligned} -a&=\tfrac{1}{2}\\ -b&=\tfrac{1}{8}\\ -c&=\tfrac{1}{16}\\ -d&=\tfrac{5}{128} \end{aligned}\right.$$

Therefore $1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}$ &c. will be the square root required; since equal squares have equal sides.

EX-

EXAMPLE VIII.

250. *To find the area of the segment* OBMP *of a circle.* Fig. 28.

Let the radius OA be unity, $OP = x$; then will * $PM = \sqrt{1 - xx}$, by the property of the circle; or $PM = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}$ &c. Therefore, by the * second rule, $x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \frac{5x^9}{1152}$ — &c. will express the area of the segment required. * art. 117. * art. 233.

COR. I.

251. Because the triangle OPM is * $= \frac{1}{2} OP \times PM$; and $PM = 1 - \frac{x^2}{2} - \frac{x^4}{8}$ — &c. that triangle will be $= \frac{x}{2} - \frac{x^3}{4} - \frac{x^5}{16} - \frac{x^7}{32} - \frac{5x^9}{256}$ &c. and this triangle subtracted from the segment OBMP, gives $\frac{x}{2} + \frac{x^3}{12} + \frac{3x^5}{80} + \frac{35x^9}{2304}$ &c. for the sector OBM. * art. 234.

COR. II.

252. The sector OBM is equal * to half the rectangle made of the radius OB, and the arc BM; if $BM = z$, then the sector OBM will be equal to $\frac{z}{2}$: Therefore, by multiplying the * art 175.

value

value above of the ſector by 2, we ſhall have $z = x + \frac{x^3}{6} + \frac{3x^5}{40} + \frac{5x^7}{112} + \frac{35x^9}{1152} +$ &c. or if we make $A = x$, $B = \frac{xx}{2}A$, $C = \frac{3xx}{4}B$, $D = \frac{5xx}{6}C$, $E = \frac{7xx}{8}D$; then will $z = A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D + \frac{1}{9}E +$ &c.

If the arc BM is 30 degrees, or the ſixth part of the ſemi-circumference, then will $OP = \frac{1}{2} = .5$. and $A = .5$, $B = \frac{A}{8}$, $C = \frac{3}{16}B$, $D = \frac{5}{24}C$, $E = \frac{7}{32}D$, $F = \frac{9}{40}E$, $G = \frac{11}{48}F$, theſe values being reduced into decimals give

$A = .5$	.50000,000
$B = .0625$	2083,334
$C = .01171,875$	234,375
$D = .00244,141$	34,877
$E = .00053,406$	5,934
$F = .00012,016$	1,093
$G = .00002,754$	212
$H = .00000,639$	43
$I = .00000,149$	9
	.52359,877

This being the value of the ſixth part of the ſemi-circumference, and therefore ſix times that value, which is 3.14159,262, will expreſs the ſemi-circumference whoſe radius is unity; true excepting the laſt figure which ſhould be 5. Conſequently, unity is to 3.14159, &c. as the radius is

is to the ſemi-circumference, or as the diameter to the whole circumference.

If the diameter be 7, then the circumference will be 21.99, &c. or 22, nearly, which is *Archimed*'s proportion: But if the diameter be ſuppoſed 113, the circumference will be 354.9999, or 355 exceeding nearly, which is the proportion of *Snellius*.

COR. III.

253. If a expreſſes the diameter of a circle, c its circumference, then, *firſt*, the ſquare aa of the diameter will be to the * area $\frac{ac}{4}$, of the circle as a to $\frac{c}{4}$, or as $4a : c$; that is, as 452 to 355. * *art.* 174.

Secondly, The cube a^3 of the diameter of a ſphere is to the * ſolid content $\frac{aac}{6}$ of the ſphere as a to $\frac{c}{6}$, or as $6a$ to c; that is, as 678 to 355. * *art.* 246.

Thirdly, The cube a^3, of the axis is to the circumſcribed cylinder * $\frac{aac}{4}$, as a to $\frac{c}{4}$, or as $4a$ to c; that is, as 452 to 355; which is the ſame proportion as that of the ſquare of the diameter to the area of the circle. * *art.* 246.

Fourthly, The cube a^3 of the axis of the inſcribed cone, will be to the cone * $\frac{aac}{12}$, as a to $\frac{c}{12}$, or as $12a$ to c; that is, as 1356 to 355. * *art.* 239.

METHOD

METHOD *for computing the Number of Shot contained in a* PILE.

SHOT are piled generally in two different manners, the base is either a square or a rectangle; the one is called a square, and the other an oblong Pile.

Fig. 29. It is evident, that the first or upper range in a square pile A B C D, contains one shot, the second 4, the third 9, the fourth 16, and so on: Therefore the number of shot contained in a square pile is found in the same manner as the sums of the squares 1, 4, 16, 25, 36, &c. of the natural numbers.

Fig. 30, I. If we suppose a triangular prism A B D C, of shot, whose height B D, or A C, being equal to the side A B, of the triangular base, to be cut diagonally C F; it is evident, that the triangu-
art. 192 lar pyramid C F D is * one third of that prism, and therefore the remainder C F A will be the two other thirds of that prism.

Now because the ranges of the triangular base are expressed by the terms of an arithmetical progression, such as 1, 2, 3, 4, 5, whence, if the number of shot in the side A F is expressed by z, then the sum of all the terms in this progression will be equal to the sum of the first and last term
* *art.* 51. $z + 1$, multiplied by * half the sum z, of the terms; therefore this triangle is equal to $\frac{z \times z + 1}{2}$.

Now the base ABC being multiplied by the number z, of these triangles contained in the prism ABDC, we shall have $\frac{zz \times z + 1}{2}$, or $\frac{z^3 + zz}{2}$, for the

contents

content of that prism; the two thirds of which $\frac{z^3+zz}{3}$, will express the part A F C.

But it is plain, that the section C F cuts the triangular range C B F into two parts; the one C B F will, by the same reason as above, be one third of that range; but $\frac{zz+z}{2}$ has been found to express a triangular range; therefore $\frac{zz+z}{2\times 3}$ will be equal to the part C B F, which, consequently, must be added to the part A C F, in order to get the sum $\frac{z^3+zz}{3}+\frac{zz+z}{2\times 3}$, or $\frac{2+z^3\ 3zz+z}{2\times 3}$, of the square pile A B C. This expression may be reduced to $\frac{z\times z+1\times 2z+1.}{2\times 3}$

EXAMPLE I.

If the base A C of the square pile A B C contains 20 shot, then $z=20$, $z+1=21$, $2z+1=41$; therefore $\frac{20\times 21\times 41}{2\times 3}=10\times 7\times 41=2870$, will be the number of shot contained in this pile.

EXAMPLE II.

Let the side A B of the oblong pile contain 10 shot, and the upper range B D, 21. then $\frac{10\times 11\times 21}{2\times 3}=5\times 11\times 7=385$, will be the number contained in the part A B C; and $\frac{zz+z}{2}=\frac{10\times 11}{2}=55$, in the triangular base; and as there

are

are 20 such triangles more in the part CBDE, $20 \times 55 =$, 1100 will be their sum; therefore $385 + 1100 = 1485$, will be the number of shot contained in the oblong pile ABDE.

This method furnishes also a general one to find the number of shot contained in any square, oblong, finished or unfinished pile.

II. For, if the sides of the upper base of an unfinished pile be m and n, it is evident that the numbers of shot contained in the several ranges, are expressed by the terms of this progression, mn, $m+1 \times n+1$, $m+2 \times n+2$, $m+3 \times n+3$, $m+4 \times n+4$, &c. which, when multiplied, give, by supposing $m+n=a$.

$$\begin{array}{l} mn \\ mn+1a+1 \\ mn+2a+4 \\ mn+3a+9 \\ mn+4a+16 \\ \&c. \quad \&c. \quad \&c. \end{array}$$

Hence, if z be the number of terms or ranges, it is plain that mnz will express the sum of the first vertical column; and $\frac{az \times z+1}{2}$, would express the sum of the second column, if z did express their number; but $z-1$ expresses the number of terms; so if we write $z-1$ for z, and z for $z+1$, we shall have $\frac{az \times z-1}{2}$ for the sum of the second column; the sum of the third would be expressed by $\frac{z \times z+1 \times 2z+1}{2 \times 3}$, if z did express the number of terms, but this number is $z-1$; if therefore

fore we write $z-1$ for z; z for $z+1$, and $2z-1$ for $2z+1$, we shall have $\frac{z \times z-1 \times 2z-1}{2 \times 3}$, for the sum of the third column. Consequently, $mnz+\frac{az \times z-1}{2}+\frac{z \times z-1 \times 2z-1}{2 \times 3}$, will express the whole sum, or the number of shot contained in this pile.

GENERAL RULE.

1. Multiply the upper sides together.
2. Take half the product of the sum of the upper sides and the corner row less one.
3. Take the sixth part of the corner row less one multiplied by twice that row less one.
4. Add these three products together, and multiply their sum by the corner row, the product will be the required number.

Example I. Let the sides of the upper surface be 20 and 4, and the corner row 6; then 20 by 4 gives - - - - - - - - - - 80
Half the sum 12 of 20 and 4, by 5 - - 60
And sixth part of 5 by 11 - - - - - - $09\frac{1}{6}$

$149\frac{1}{6}$

This sum multiplied by 6 gives 895 for the number of shot contained in this unfinished oblong pile.

Examp. II. Let there be a square finished pile whose corner row is 6, then 1 by 1 gives - - 1
Half the product 10 of 2 by 5 - - - - - 5
And the sixth part of 5 by 11 - - - - $9\frac{1}{6}$

$15\frac{1}{6}$

This sum multiplied by 6 gives 91 for the number sought.

Ex-

Examp. III. Let the pile be ſquare and unfiniſhed, whoſe upper ſides being 6, and the corner row 10; then 4 by 4 gives - - - 16
Half the product 72 of 8 by 9 - - - 36
And the ſixth part of 9 by 19 is - - - $28\frac{1}{2}$

$80\frac{1}{2}$

This ſum multiplied by 10 gives 850 for the number ſought.

Examp. IV. Let the pile be oblong and finiſhed, whoſe upper row being 20 and the corner one 12, then the product of 20 by 1 gives - - 20
Half the product 231 of 21 by 11 - - $115\frac{1}{1}$
And the ſixth part of 11 by 23 - - - $42\frac{1}{6}$

$177\frac{2}{3}$

This multiplied by 12 gives 2132 for the number required.

Theſe examples are ſufficient to ſhew the extent of this rule, and the eaſy manner of computing piles of ſhot, let them be ſquare, oblong, finiſhed, or unfiniſhed; for which reaſon we ſhall conclude this firſt book, by adviſing the learner to make the contents of it as familiar as poſſible; for it contains the foundation of all the other branches of the mathematics.

BOOK

BOOK II.

Of CONIC-SECTIONS.

SECT. I.

Of the PARABOLA.

DEFINITIONS.

Plate IX. *Fig.* 1.

1. HAVING placed a ruler E E, on a plane, and a a ſquare L D *f* ſo as one of its ſides D L touches the ruler, and a thread F M *f*, equal in length to the other ſide D *f* of the ſquare being faſtened with one end, to the end *f*, of the ruler, and the other end to any point fix F in that plane on the ſame ſide of the ruler as the ſquare.

Then if the ſide D L ſlides along the ruler E E, whilſt with a pin M, the part M *f* of the thread is always kept tight and quite cloſe to the edge of the ſquare; the pin M will deſcribe the part A M of a curveline.

If the ſquare is placed on the other ſide of the point, fix F; you may deſcribe in the ſame manner, the part A m, of a curveline; and the whole curve M A m is called a Parabola.

2. The point fix F is called the Focus.

3. The line A Q, drawn through the focus F perpendicular to the ruler E E, the Axis.

4. The line repreſented by the edge of the ruler E E, is called the Directrix.

5. A line double the diſtance B F, of the ruler from the focus, the Parameter.

6. And the part A P, of the axis, terminated by any perpendicular Abſciſſa.

7. The point A where the curve meets the axis, the Vertex.

8. A line which biſects ſeveral parallel lines, terminated by any conic-ſection, is called a Diameter; and the parallels the Ordinates to that diameter; though their halves are commonly called ſo.

9. A right line which touches a conic-ſection but in one point, and when produced both ways, does not fall within the curve, a Tangent.

COR. I.

254. It follows from the definition of the parabola, that if from any of its points M, there be drawn two lines, the one to the focus F, and the other perpendicular to the directrix, they will be equal: For by taking from the length of the ſide *f* D of the ſquare, and from the thread *f* M F its equal, the part *f* M, which is common to both, it is plain that the remainders M D and M F are equal.

COR. II.

255. It follows alſo, that the diſtance F A between the focus and vertex, is equal to the diſtance A B, between the vertex and ruler. For ſuppoſe the ſquare to fall on the axis A Q; then becauſe M F is always equal to M D, when M D becomes A B, its equal M F, becomes A F. Conſequently, A F = A B.

THE-

THEOREM I.

256. *The square of any perpendicular* MP *to the axis, is equal to the rectangle made of the parameter and the corresponding abscissa* AP. *I say,* $4AF \times AP = \overline{PM}^2$.

For * $FM = BP = AP + AF$; and $FP = AP - AF$; therefore $FM + FP = 2AP$, and $FM - FP = 2AF$. But because of the right angled triangle FPM, the difference between the squares of FM, FP, is equal * to the square of PM, as likewise to the rectangle of the sum 2AP and difference 2AF * of the sides FM and FP. Consequently, $2AP \times 2AF$, or $4AF \times AP = \overline{PM}^2$. * *art.* 254. * *art.* 49. * *art.* 46.

If we call the parameter p; then $4AF = p$, and $p \times AP = \overline{PM}^2$.

COR. I.

257. Since the square of any perpendicular to the axis is always equal to the rectangle made of the parameter p, and the corresponding abscissa, it follows that if NQ be drawn perpendicular to the axis, then will $p \times AQ = \overline{QN}^2$. Consequently, $\overline{PM}^2 : \overline{QN}^2 :: (p \times AP : p \times AQ) :: AP : AQ$. Which shews, that the squares of the perpendiculars to the axis are always to each other as their corresponding abscissas.

COR. II.

258. It is evident, that the parabola opens continually as it recedes from the vertex A; for as the abscissa AP increases, so increases also the square of the corresponding perpendicular PM.

COR. III.

259. It is no less evident, that the axis bisects

the parabola, for ſince the perpendiculars P M, at equal diſtances, A P from the vertex A, are always equal on both ſides the axis, the part of the parabola on one ſide of the axis is equal to the other. Conſequently, the axis is a *diameter* which biſects its ordinates at right angles.

PROB. I.

Fig. 2. 260. *To draw a tangent to any point* M *of the parabola.*

Draw M L parallel to P A, and equal to the diſtance M F, from the point M to the focus F; join F L, then the line M D, which is at right angles to the line F L will be the tangent required.

*art. 254. Let L E be perpendicular to A P; then * M L is = E P = M F by conſtruction: And if from any other point m in D M, a line m l is drawn parallel to M L, and another to the focus F, it

* art. 15. is evident that the line m L is equal * to m F: Now becauſe of the right angled triangle m l L, the hypothenuſe m L is always greater than the ſide m l; but this line ſhould be * equal to m F,

* art. 254 if the point m were in the curve; therefore the line M D touches the parabola only in the given point M.

COR. I.

261. If the tangent produced meet the axis in T, then becauſe the angles F M T, L M T, are

* art. 16. * equal, as likewiſe the alternate ones F T M, T M L, the angles F M T, F T M, are equal;

* art. 26. and hence the ſides F M and F T * are equal;

* art. 254. but * F M = E P, and therefore F T = E P,

* art. 255. and becauſe * A F = A E, we have A T = A P. Conſequently, the ſubtangent P T is double the correſponding abſciſſa A P.

COR.

COR. II.

262. If MK be perpendicular to the tangent at M, then becauſe of the parallels MK, LF, we have ML = FK = *EP; and taking away the common line FP, we ſhall have PK = EF; that is, the part PK, terminated by MP and the perpendicular MK to the tangent, is equal to half * the parameter *p*. *art. 254. *art. 256.

COR. III.

263. The right angled ſimilar triangles TPM, TMK, give TP or 2AP : TM :: TM : TK = 2FK = 2FM; and therefore AP : TM :: TM : 4FM: Since the rectangles of the means, and thoſe of the extremes, are the ſame in both theſe proportions.

LEMMA.

264. *If from any point* M *in the parabola, there be drawn a tangent* MT, *an ordinate* MP *to the axis* AQ, *and a line* MR, *parallel to the axis; and if from any other point* L, *an ordinate* LQ *be drawn, meeting* MR *in* R, *and a line* LK *parallel to the tangent* MT, *meeting* MR *in* E, *and the axis in* V; *then will the triangle* ERL *be always equal to the correſponding parallelogram* TMEV. Fig. 3.

For the ſimilar triangles TPM, VQL, are as the ſquares of their homologous * ſides PM, QL; and theſe ſquares are as * the correſponding abſciſſas, *viz.* ½TP, or * AP : AQ :: TMP : VQL, and the triangle TMP, is to the ſpace TMRQ, as AP : AQ :: TMP : TMRQ; therefore VQL = TMRQ by equality of * ratios; and taking away the common ſpace VERQ, we ſhall have TMEV = ERL. * art. 88. * art. 257. * art. 261. * art. 53.

THE-

THEOREM II.

265. *The ſquare of any line* E L *parallel to the tangent* T M, *will be to the ſquare of that tangent, as the part* M E *of the line* M R *is to* A P, *or* A T.

For the ſimilar triangles T P M, E R L, are
*art. 88. as the ſquares of their homologous * ſides $\overline{TM}^2 : \overline{EL}^2 :: TMP : ELR$; and the triangle T P M is to the parallelogram T M E V, as $\frac{1}{2}$ A T, or
*art. 264. A P : M E :: T M P : T M E V; but * TMEV $=$ E L R; therefore A P : M E :: $\overline{TM}^2 : \overline{EL}^2$ by equality of ratios.

COR. I.

266. If from the point M a line be drawn to
art. 263. the focus F, then becauſe A P : T M : : T M : 4 F M, or $\overline{TM}^2 = 4AP \times FM$; this value being ſubſtituted into the laſt proportion, gives A P : M E :: $4AP \times FM : \overline{EL}^2$, or 1 : M E : : 4 F M : $\overline{EL}^2 = 4FM \times ME$.

COR. II.

267. If the line L E produced, meets the parabola in K, it is evident that the ſquare of E K is alſo equal to the rectangle $p \times$ M E; and therefore $\overline{EL}^2 = \overline{EK}^2$, or E L $=$ E K; for whether the triangle E L R be placed on one Side or the other of the line M R, the demonſtration in art. 265 will be the ſame. Conſequently, the line M R is a diameter, and the lines L K, parallel to the tangent at M, are its ordinates: *Which ſhews that the diameters are parallel to the axis in the parabola.*

COR. III.

268. If we call 4 F M, p; then becauſe $p \times$ M E $= \overline{EL}^2$;

$= \overline{EL}^2$; it is evident that whether M E be a diameter or the axis, the ſquare of any ordinate E L is always equal to the rectangle made of the correſponding abſciſſa M E and the parameter p; and the parameter of any diameter or axis is always equal to four times the diſtance M F of the vertex of the diameter or axis from the focus F.

COR. IV.

269. If another ordinate N B be drawn to the diameter M R, then will $p \times MB = \overline{NB}^2$; therefore $p \times ME : p \times MB$; or, $ME : MB :: \overline{LE}^2 : \overline{NB}^2$. Which ſhews that the ſquares of the ordinates, to any diameter or axis, are to each other as the correſponding abſciſſas.

COR. V.

270. If from any points N, L, in the parabola, lines N C, L D, are drawn parallel to the diameter, meeting the tangent in C, D, the ſquares of the parts M C, M D, of the tangent terminated by theſe lines will be to each other as the lines C N, D L; for M C = B N, M D = E L, and C N = M B, D L = M E.

THEOREM III.

271. *If a line* L l *interſects a diameter* M R, *either within or without the parabola, the rectangle of its parts terminated by that diameter, will be equal to the rectangle made of the part* M R *of the diameter terminated by the line, and the parameter* p, *of the diameter* A Q, *to which that line is an ordinate*, viz. $p \times MR = LR \times Rl$. Fig. 4.

From the vertex M, of the diameter, draw the ordinate M P to the diameter A Q; then * $p \times AP = \overline{PM}^2$, and $p \times AQ = \overline{QL}^2$; and * art. 268.

therefore $p \times AQ - p \times AP$, or $p \times MR = \overline{QL}^2 - \overline{PM}^2$: But the difference between two squares is equal to the rectangle of the * sum and difference of their sides; and because * $QL = Ql$, there will be $Rl = QL + PM$, and $RL = QL - PM$. Consequently, $p \times MR = LR \times Rl$.

* *art.* 46.
* *art.* 267

COR. I.

272. Hence, if another line Kk be drawn parallel to Ll, intersecting the diameter MR in S, then will * $p \times MS = KS \times Sk$: Consequently, $p \times MS : p \times MR$, or $MS : MR :: KS \times Sk : LR \times Rl$. *Which shews that if any diameter intersects two parallel lines, either within or without the parabola, the rectangles of the parts of these lines, terminated by the diameter, are as the parts of that diameter, terminated by these lines.*.

* *art.* 271.

COR. II.

273. It follows also, that if any line mn intersects the diameter MR in the same point S as the line Kk, and q be the parameter of the diameter to which mn is an ordinate, then will $q \times MS = mS \times Sn$. Therefore $q \times MS : p \times MS$, or $q : p :: mS \times Sn : KS \times Sk$.

COR. III.

274. If the line mn, produced if necessary, meets Ll in D; then will *(q : p ::)* $mS \times Sn : KS \times Sk :: nD \times Dm : LD \times Dl$. In general, the rectangles of the parts of any two parallel lines terminated by the parabola, and any other line, will always be as the rectangles of the parts of that line terminated by the parabola and the parallel lines, whether they meet or intersect each other within or without the parabola.

PROB.

PROB. II.

275. *To find the area of a parabolic ſpace* AMP, *terminated by any ordinate* PM *to the axis* AP. Fig. 5.

Draw AQ perpendicular, and MQ parallel to the axis; then if $AP = x$, $PM = y$, and the parameter unity; then * will $x = yy$: Now since $AQ = PM = y$, and $QM = x = yy$, the area AMQ, whoſe perpendicular QM is expreſſed by the ſquare yy, of the baſe AQ, will be * $\frac{y^3}{3}$, or becauſe $x = yy$, $= \frac{xy}{3}$; that is, the external ſpace AMQ is the third part of the rectangle PQ; therefore the internal ſpace AMP is the other two thirds of that rectangle. * art. 256. * art. 233.

COR. I.

276. If c be the circumference whoſe radius is unity, then becauſe the circumferences are * as their radii, $1 : PM\ (y) :: c : cy =$ to the circumference of the radius PM; and * $\frac{cyy}{2}$, the area of the circle deſcribed in the rotation of the parabola about the axis AP; or becauſe $x = yy$; that circle will be expreſſed by $\frac{cx}{2}$; therefore the ſolid deſcribed by AMP in that rotation, will * be $\frac{cxx}{4}$, or becauſe $= yy$, it will be $= \frac{cxyy}{4}$. * art. 171. * art. 174. * art. 233.

But as the cylinder deſcribed by the rectangle PQ in that rotation, is * $\frac{cyyx}{2}$; it is evident,

that

that the ſolid deſcribed by the inward ſpace A M P is half that cylinder; and ſo the ſolid deſcribed by the outward ſpace A M Q the other half, which conſequently is equal to the former.

COR. II.

277. Since * 1 : Q M (x) :: c : cx = to the circumference of the radius Q M, and $\frac{cxx}{2}$; or becauſe $x = yy$, $\frac{cy^4}{2}$ will be the area deſcribed by the line Q M in the rotation of the parabola about A Q; therefore * $\frac{cy^5}{10}$, or its equal $\frac{cxxy}{10}$, will expreſs the ſolid content of the ſolid deſcribed by the outward ſpace A M Q; this being the fifth part of the cylinder $\frac{cxxy}{2}$, deſcribed by the rectangle P Q, about A Q; the ſolid deſcribed by the inward ſpace A M P, about that line, will be the four-fifth part of that cylinder.

* *art.* 171.

* *art.* 233.

SECT. II.

Of the ELLIPSIS.

DEFINITIONS.

Fig. 6. 8. HAVING faſtened the two ends of a thread F M *f*, to two points, fix F, *f*, in a plane in ſuch a manner as the interval F *f*, between theſe points, be leſs than the length of the thread; then if with a pin M the thread is always

always kept tight, by moving this pin round the points fix, it will describe a curve line M A N a, which is called an Ellipsis.

2. The points fix F, *f*, are called the Foci.

3. The line A a, which passes through the foci F, *f*, and is terminated by the ellipsis, is called the first or great Axis.

4. The point C, which bisects the axis, is called the Center.

5. The line B b, drawn through the center C at right angles to the first axis A a, and terminated by the ellipsis, is called the second or little Axis.

6. The line which is a third continued proportion to half the first and second axes, is called the Parameter to the first axis.

7. A diameter which is parallel to the ordinates of another diameter, is called its Conjugate.

COR. I.

278. It follows, from the first definition, that if from any point M in the ellipsis, there be drawn two lines to the foci F, *f*, their sum will always be the same, let that point be taken where it will.

COR. II.

279. When the point M falls at A, the part F M of the thread will become $= A F$, and the other part $f M = f A$; so that $f A + F A$ is equal to the length of the thread; and when the point M falls on a, the part $f M$ becomes $= f a$, and the part $F M$, $= F a$; therefore $f A + F A = F a + f a$, and by taking away the common line F *f*, we have $2 A F = 2 a f$, or $A F = a f$.

COR.

COR. III.

280. Hence, the first axis A a is equal to the thread fM F; for because fA + FA is equal to it, and FA = fa; we have fA + FA = fA + fa = Aa.

COR. IV.

281. The distances of the foci, f, F, from the center C are equal; for because CA = Ca by defin. 4, and AF = af; CF will be = Cf.

COR. V.

282. The lines drawn from either of the foci f, to the extremities of the second axis B, b, are each equal to half CA the first axis: For it is clear that when the point M falls on B or b, the two right angled triangles FCB, fCB, having the sides fC, CF, equal, and CB common to both, have their * hypothenuses FB, fB, equal; that is, FM becomes equal to fM, and their sum being equal to the first axis, their half FB, or fB, will be = CA. Hence also,

* art. 15.

I. The second axis Bb is bisected by the center C; for since fB = fb, and Cf common to both right angled triangles BCf, bCf, they * are equal in all respects. Therefore BC = Cb.

* art. 15.

II. The second axis Bb is always less than the first Aa; for since FB = CA, the hypothenuse FB will be always greater than any one of the sides BC. Therefore 2BC or Bb is less than 2CA or Aa; and when the axes become equal, the ellipsis becomes a circle.

III. The square of half the greater axis is equal to the sums of the squares of half the lesser CB, and

and that of the distance C F, of the focus from the center.

IV. The difference between the squares of F B half the greater axis and the distance C F, of the focus from the center, is equal to the square of half C B the second axis; or because F a = F B + F C, and F A = F B — F C, we have * A F × F a = $\overline{B C}^2$. * art. 46.

THEOREM I.

283. *If from any point* M *a perpendicular* M P *be drawn to the first axis* A a, *and* C X *be taken a fourth proportional to* C A, C F, *and* C P, *then will* A X = F M, *and* a X = M *f*.

Because in the triangle F M *f* the rectangle of the sum and difference of the sides F M, M *f*, is * equal to the rectangle of the sum and difference of the segments F P, P *f*; but * 2 C A is the sum of the sides F M, *f* M, if D is half their difference; then as 2 C F = *f* F, and 2 C P = *f* P — P F. Therefore 2 C A × 2 D = 2 C F × 2 C P; or C A : C F :: C P : D = C X by supposition. * art. 46. * art. 280.

Therefore, since A C is * half the sum of the lines F M and *f* M, and C X half the difference, A X will be equal to the lesser F M, and a X equal to the greater *f* M. * art. 280.

THEOREM II.

284. *The square of the perpendicular* M P, *drawn from any point in the ellipsis to the first axis, is to the rectangle of the corresponding parts of that axis as the square of half the second axis is to the square of half the first. I say,* $\overline{P M}^2$: A P a :: $\overline{C B}^2$: $\overline{C A}^2$.

Because

* art. 283. * art. 280. Becauſe * CX or * CA—FM : CP :: CF : CA, by adding the antecedents to their conſequents we get aP—FM : aF :: CP : CA, and ſubtracting the antecedents from their conſequents I. FM+FP : AP :: aF : CA.

Now if in the firſt proportion the antecedents are ſubtracted from their conſequents, there will be FM—AP : CP :: AF : CA; and by adding antecedents to antecedents, and conſequents to conſequents, II. FM—FP : aP :: AF : CA.

If the terms of the proportion I. are multiplied by the reſpective terms of the proportion II. and * art. 46. we take the ſquare of PM, inſtead of the rectangle * made of the ſum and difference of the ſides * art. 282. FM and FP of the right angled triangle FPM, as likewiſe the ſquare of CB inſtead of the * rectangle AF × Fa; we ſhall have $\overline{PM}^2 : APa :: \overline{CB}^2 : \overline{CA}^2$.

COR. I.

285. Since the ſquare of any perpendicular PM is always to the rectangle of the correſponding parts AP, Pa, of the axis, as the ſquare of CB to the ſquare of CA; when the perpendicular paſſes through the focus F, as LF, then the above proportion becomes $\overline{LF}^2 : AFa :: \overline{CB}^2 : \overline{CA}^2$: * art. 282. or becauſe AFa * is $= \overline{CB}^2$; we have $L\overline{F}^3$: * art. 82. $\overline{CB}^2 :: \overline{CB}^2 : \overline{CA}^2$, or * LF : CB :: CB : CA; therefore LF is the parameter of CA.

COR II.

* art. 176. 286. Hence, becauſe * $\overline{AC}^2 : \overline{CB}^2 :: CA : LF$: Therefore $\overline{PM}^2 : APa :: LF : CA$, * art. 282. by equality of ratios. *Which ſhews that the ſquare of any perpendicular*

perpendicular to the first axis is to the rectangle of the corresponding parts of the axis as the parameter L F *is to half the first axis.*

COR. III.

287. If there be drawn another perpendicular N Q to the first axis, then will $\overline{QN}^2 : AQa :: LF : CA$; and as $\overline{PM}^2 : APa :: LF : CA$, it is evident that $\overline{PM}^2 : APa :: \overline{QN}^2 : AQa$, by equality; or $\overline{PM}^2 : Q\overline{N} :: APa : AQa$, by alteration. Consequently, the squares of the perpendiculars to the first axis, are as the rectangles of the corresponding parts of the axis.

COR. IV.

288. It is manifest that the perpendiculars on either side the first axis, which are equidistant from the center, are equal; for is $CP = CQ$, then will $AP = aQ$, and $AQ = aP$; therefore $APa = AQa$, and $\overline{PM}^2 = \overline{QN}^2$, or $PM = QN$. Consequently, the axis is also a diameter, and the perpendiculars P M, Q N, to it, its ordinates.

COR. V.

289. If from any point M in the ellipsis, a line M R be drawn perpendicular to the second axis B b; then because C R = P M, and R M = C P; and since the rectangle A P a is equal to the difference * $\overline{CA}^2 - \overline{CP}^2$ between the squares of half C A the axis, and the part C P; therefore $\overline{PM}^2 : \overline{CA}^2 - \overline{CP}^2 :: \overline{CB}^2 : \overline{CA}^2$; or by substituting for C P, P M, their equals R M, C R, we shall have $\overline{CR}^2 : \overline{CA}^2 - \overline{RM}^2 :: \overline{CB}^2 : \overline{OA}^2$; *Fig.* 8.

* *art.* 46.

$\overline{CA}^2$; and comparing the difference between the consequents to that of the antecedents, to the terms of the last ratio, we shall have $\overline{RM}^2$: BRb :: $\overline{CA}^2$: $\overline{CB}^2$; because $\overline{CB}^2 - \overline{CR}^2 =$ BRb. *Which shews, that the second axis is the conjugate diameter to the first axis, and that the perpendiculars* MR, *to that axis, are its ordinates.*

COR. VI.

290. Therefore, since the square of any ordinate MR to the second axis, is to the rectangle of the corresponding parts of that axis, as the square of half the first is to the square of half the second, the properties of this axis, in respect to its ordinates, are the same as those of the first. Hence,

I. The lines drawn through the extremities of either axes, parallel to the other, touch the ellipsis in those points, for because the ordinates on both sides of the same axis are constantly equal, the lines parallel to them fall entirely without the ellipsis.

II. Each axis divides the ellipsis into two equal parts, so that if the one half was laid over the other, it would agree with it in every point.

PROB. I.

Fig. 7. 291. *To draw a line* MD *which shall touch the ellipsis in any given point* M.

From the foci *f*, F, draw lines to the given point M; in *f*M produced take ML = MF; then the line MD perpendicular to FL will be the tangent required.

From any other point m in MD, draw lines to the foci and to the point L; then because Lm + m*f*, or its equal Fm + m*f*, is always greater than

than * fL or * Aa; and as the sum of these lines drawn from any point in the ellipsis to the foci, is * always equal to Aa, it follows that the point m is not in the ellipsis; and consequently MD touches the ellipsis in the only point M. * art. 23. * art. 280. * art. 280.

COR. I.

292. If the tangent produced both ways meets the axes in T, t, it is manifest that the angles fMt, FMT, made by the lines drawn from the foci to the point of contact M and the tangent, are equal.

For since $ML = MF$ by construction, and MD perpendicular to FL, the angles LMD, FMD, are * equal; and the angle LMD is equal to its opposite one fMt. Consequently, the angles FMD, fMt, are equal. * art. 18.

THEOREM III.

293. *If from the point of contact* M, *an ordinate* MP *be drawn to the first axis, half this axis will be a mean proportional, between the distances, of this ordinate, and the point* T *where the tangent meets the axis, from the center. I say,* $\therefore CP : CA : CT$.

Draw CD; then because $CF = Cf$, and * $FD = DL$, the line CD will be parallel to fL and equal to half of it, or * equal to $CA = CD$; and because of the parallels fM, CD, we have $fM : CD$, or $CA :: fT : CT$; or $fM - CA : fC :: CA : CT$. And if AX be taken equal to FM, then will * $fM = aX$, or $fM - Ca = CX$; but because * $CX : CF :: CP : CA$, we have $CP : CA :: CA : CT$, by equality of ratios: Since $CA = Ca$, and $Cf = CF$. * art. 291. * art 280. * art. 283. * art. 283.

COR. I.

294. If the line Kk, drawn perpendicular to the tangent at M, meets the first axis in K and the second in k; then because of the parallels LF, MK, and fM, CD, we have CD or CA : CF :: fM : fK; or subtracting the terms of the last ratio from those of the first, CA : CF :: fM — CA : CK; but * CA : CF :: CP : CX: Therefore, multiplying the terms of the first ratio by the respective terms of the second, then will $\overline{CA}^2 : \overline{CF}^2 :: CP : CK$; because fM — CA = CX.

* art. 283.

* art. 283.

COR. II.

295. Hence, since $\overline{CA}^2 : \overline{CF}^2 :: CP : CK$, we get by subtraction $\overline{CA}^2 : \overline{CA}^2 - \overline{CF}^2 :: CP : PK$; or because * $\overline{CA}^2 - \overline{CF}^2 = \overline{CB}^2$; we shall have $\overline{CA}^2 : \overline{CB}^2 : CP : PK$: And by the similarity of triangles CP or pM : PK :: pk : PM. Therefore $\overline{CA}^2 : \overline{CB}^2 :: pk : PM$, by equality of ratios; or, $\overline{CB}^2 : \overline{CA}^2 :: Cp : pk$.

* art. 282.

COR. III.

296. Because * CP × CT : PK × CT :: CP : PK, we have $\overline{CA}^2 : \overline{CB}^2 :: CP \times CT : PK \times CT$ by equality of ratios: But since * CP × CT = $\overline{CA}^2$; we have also $\overline{CB}^2 = PK \times CT$; or, PK : CB :: CB : CT. Therefore the similar triangles MPK, TCt, give PM : PK :: CT : Ct. Consequently, * PM or Cp : CB :: CB : Ct. Which shews that the properties of the tangents are the same in respect to both axes.

* art. 65.

* art 293.

* art. 85.

THE-

THEOREM IV.

297. *Any line* M N *passing through the center* C, *and being terminated by the ellipsis will be bisected by the center. I say*, C M = C N. Fig. 8.

By drawing M P, N Q, perpendicular to the axis A a, the right angled triangles C P M, C Q N, will be similar, whence if C P = C Q, then will P M = Q N, and C M = C N; but since the ordinates which are equidistant from the center are equal, it is evident that if the point M is in the ellipsis, the point N is also in the ellipsis. Consequently M N is bisected by the center.

LEMMA.

298. *If from the extremity* L *of any ordinate* L Q *to the first axis a line* L V *be drawn parallel to the tangent* M T, *and from the point* M *of contact, a line* M m *through the center, intersecting* M V *in* E, *and the ordinate* L Q, *produced if necessary, in* R; *the quadrilateral* T M E V *will be always equal to the corresponding triangle* L E R. *I say*, T M E V = L E R. Plate X. Fig. 9.

The similar triangles T P M, V Q L, are as the squares * of their homologous sides P M, Q L, or * T P M : V Q L :: $\overline{C A}^2 - \overline{C P}^2 : \overline{C A}^2 - \overline{C Q}^2$; and the similar triangles C P M, C Q R, are also as the squares of their homologous sides C P, C Q; that is, C P M : C Q R :: $\overline{C P}^2 : \overline{C Q}^2$: If the terms of this last proportion are added to their respective ones of the former, we shall have * C M T : V E C + L E R :: $\overline{C A}^2 : \overline{C A}^2$; therefore C M T = V E C + L E R; and by subtracting the common space V E C, there will remain T M E V = L E R. * *art.* 88. * *art.* 287. * *art.* 64.

COR.

299. Since this property is always the same, wherever the line LV is drawn parallel to the same tangent; if it passes through the center, or becomes Nn, the space TMEV becomes TMC, and the triangle LER the triangle NCr. Therefore TMC = CNr.

THEOREM V.

300. *The square of any line* LE, *drawn parallel to a tangent* MT, *terminated by the line* Mm *which passes through the center and the point of contact, will be to the rectangle of the corresponding parts of the line* Mm, *as the square of* CN, *parallel to the tangent is to the square of* CM.

I say, $\overline{LE}^2 : ME\,m :: \overline{CN}^2 : \overline{CM}^2$.

For the similar triangles CEV, CMT, give * $\overline{CM}^2 : \overline{CE}^2 :: CMT : CEV$; and subtracting the consequents from their antecedents, and comparing the antecedents to the differences $\overline{CM}^2 : \overline{CM}^2 - \overline{CE}^2 :: CMT : TMEV$. But the similar triangles CNr, LER, give * $\overline{CN}^2 : \overline{LE}^2 :: CNr : LER$; and because * CMT = CNr, and * TMEV = LER, we have $\overline{LE}^2 : \overline{CN}^2 :: \overline{CM}^2 - \overline{CE}^2 : \overline{CM}^2$, by equality of ratios; because the line Mm is bisected by the center C, the rectangle MEm is equal to * $\overline{CM}^2 - \overline{CE}^2$. Consequently, $\overline{LE}^2 : ME\,m :: \overline{CN}^2 : \overline{CM}^2$, by alteration.

* *art.* 69.
* *art.* 69.
* *art.* 299.
* *art.* 298.
* *art.* 46.

COR.

Plate IX

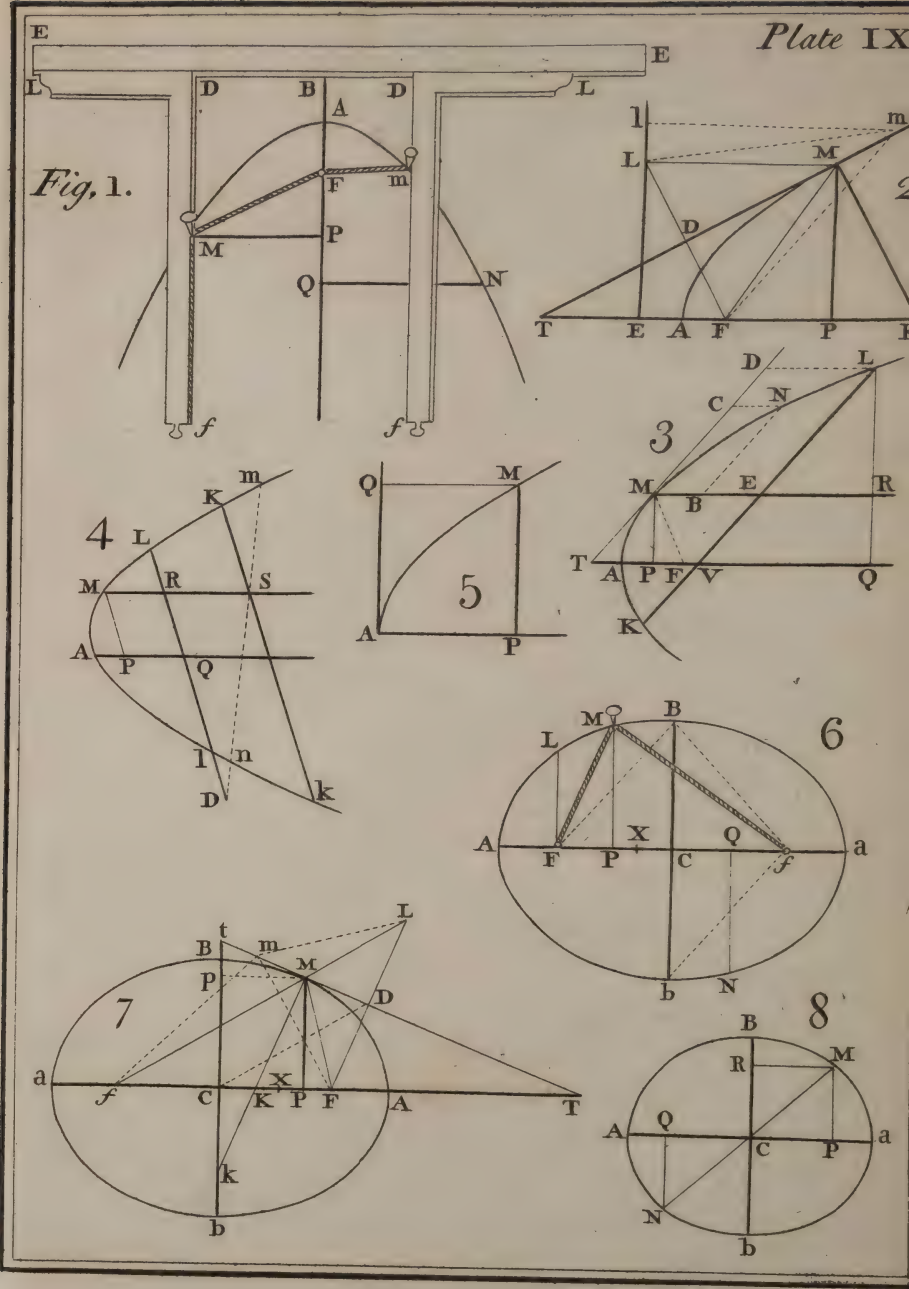
Fig. 1.
E
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COR. I.

301. Since if from any point L in the ellipſis, a line is drawn parallel to the tangent MT, the ſquare of that line LE is always to the rectangle MEm of the correſponding parts of Mm; in the given ratio, of the ſquares of CN and CM, of which ſide ſoever the point L is taken in reſpect to the line Mm; it is evident that if LE is produced, ſo as to meet the ellipſis in another point K, the line LK will be biſected by Mm: For the ſquare of KE will be to the rectangle MEm as $\overline{CN}^2$ to $\overline{CM}^2$; and ſince the ſquare of LE is to the ſame rectangle MEm, as $\overline{CN}^2$ to $\overline{CM}^2$; it follows that $\overline{KE}^2 = \overline{LE}^2$, or KE = LE. Conſequently, Mm is a diameter, and Nn its conjugate. *Which ſhews that all diameters in the ellipſis paſs through the center.*

COR. II.

302. Hence, becauſe the property of any conjugate diameters Mm, Nn, is the ſame as that of the axes: It follows,

I. That all the ordinates, equidiſtant from the center, on either ſide of the diameters, are equal.

II. That the ſquares of any two ordinates are to each other as the rectangles of the correſponding parts.

III. Laſtly, all what has been proved in reſpect to one diameter is likewiſe true in reſpect to its conjugate.

THEOREM VI.

303. *If a line* Ll *interſects any diameter* Mm, *either within or without the ellipſis, the rectangle of* Fig. 10.

its parts, terminated by the diameter, will be to the rectangle of the corresponding parts of the diameter, as the square of the semi-diameter CN, *parallel to this line, is to the square of half the diameter which it cuts. I say,* $LRl : MRm :: \overline{CN}^2 : \overline{CM}^2$.

Let Aa be the diameter to which Ll is an ordinate, draw Nu and RS parallel to the tangent MT, the first intersecting the diameter Mm in F, the rest being the same as before; then the similar triangles QRS, QLV, and CNu, give $\overline{QL}^2 : \overline{QR}^2 :: QLV : QRS$; subtracting the consequents from the antecedents, and comparing the antecedents to the respective differences $\overline{QL}^2 : \overline{QL}^2 - \overline{QR} :: QLV : RLVS$, and $\overline{QL}^2 : \overline{CN}^2 : QLV : CNu$; therefore I. $\overline{CN}^2 : \overline{QL}^2 - \overline{QR}^2 :: CNu : RLVS$, by equality of ratios.

The similar triangles CMT, CRS, give $\overline{CM}^2 : \overline{CR}^2 :: CMT : CRS$; subtracting the consequents from the antecedents, and comparing the antecedents to the respective differences II. $\overline{CM}^2 : \overline{CM}^2 - \overline{CR}^2 :: CMT : RMTS$.

* *art.* 298. But * $TMEV = LER$; by adding VERS to both, we get $RMTS = RLVS$; and since * *art.* 298. * $TMFu = CNF$; by adding CFu to both, then will $CMT = CNu$. Therefore, the two former proportions I, II, give $\overline{CN}^2 : \overline{QL}^2 - \overline{QR}^2 :: \overline{CM}^2 : \overline{CM}^2 - \overline{CR}^2$; or because* *art.* 46. $\overline{QL}^2 - \overline{QR}^2 = LRl$, and $\overline{CM}^2 - \overline{CR}^2 = MRm$; it will alternately be $LRl : MRm :: \overline{CN}^2 : \overline{CM}^2$.

COR

COR. I.

304. Hence, if through the same point R the line K k be drawn parallel to the diameter C X, then will $\overline{CM}^2 : MRm :: \overline{CN}^2 : LRl$, and $\overline{CM}^2 : MKm : \overline{CX}^2 : KRk$. Therefore by equality of ratios. $\overline{CN}^2 : LRl :: \overline{CX}^2 : KRk$. *Fig.* 11.

COR. II.

305. If the line H h, parallel to L l, intersects the diameter M m, in r, either within or without the ellipsis, then will $MRm : Mrm :: LRl : Hrh (:: \overline{CM}^2 : \overline{CN}^2.)$

COR. III.

306. If the lines K k, H h, produced if necessary, meet in D, either within or without the ellipsis, then will $KDk : HDh :: KRk : LRl$: In general, the rectangles of the parts of two parallel lines, are always to each other as the rectangles of the respective parts of any other line which intersect these parallels; and if one or two become tangents, these rectangles become squares: This is a natural Consequence, from what has been said above.

THEOREM VII.

307. *If from the point of contact* M, *an ordinate* M P *be drawn to the axis* A a, *the rectangle of the parts* C T, C P, *terminated by the tangent, the ordinate, and the center, will be equal to the rectangle of the corresponding parts of the axis terminated by the ordinate. I say*, $CP \times PT = APa$. *Fig.* 12.

Because * $CT : CA :: CA : CP$, we have by subtraction $TA : CA :: AP : CP$, by adding antecedents to antecedents. and comparing the re- *art. 293.

ſpective ſums to the terms of the ſecond ratio, we have $PT : AP :: Pa : CP$; or, $CP \times PT = APa$.

COR. I.

308. It appears alſo, that if from the extremity N of the diameter parallel to the tangent, a tangent N t, and an ordinate N Q be drawn to the ſame axis A a, the ſimilar triangles T M C, t N C, give $CT : Ct :: PM : QN$; and becauſe $\dot{=} CP : CA : CT$, as likewiſe $\dot{=} CQ : CA : Ct$, we have * $CT : Ct :: CQ : CP$. Therefore, $CQ : CP :: PM : QN$, by equality of ratios; or, $CQ \times QN = CP \times PM$.

* *art.* 86.

COR. II.

309. Hence, the ſquare of C Q is equal to the rectangle of A P a: For the ſimilar triangles T P M, C Q N, give * $TP : CQ :: PM : QN$; and ſince $CQ : CP :: PM : QN$, we have $TP : CQ :: CQ : CP$; * but $TP : AP :: Pa : CP$; therefore * $AP : CQ :: CQ : aP$; or, $APa = \overline{CQ}^2$. It may be proved in the ſame manner that $AQ : CP :: CP : Qa$; or $AQa = \overline{CP}^2$.

* *art.* 69.
* *art.* 307.
* *art.* 86.

COR. III.

310. Since * APa, or * $\overline{CQ}^2 : \overline{PM}^2 :: \overline{CA}^2 : \overline{CB}^2$, we have * $CQ : PM :: CA : CB$; and * by the ſame argument $CP : QN :: CA : CB$.

* *art.* 284.
* *art.* 309.
* *art.* 82.

THEOREM VII.

311. *The parallelogram* G H, *made by any two conjugate diameters* M m, N n, *is always equal to the rectangle made by the two axes* A a, B b.

Let the tangent H M meet the axis A a in T, draw the ordinates M P, N Q, to that axis, and from

from the center C draw C r perpendicular to the tangent T H; then the similar triangles T C r, C Q N, give C N : Q N :: C T : C r; but * C P : C A :: C A : C T, and * C P : C A :: Q N : C B; therefore Q N : C B :: C A : C T, by equality of ratios; and the first and last proportion give * C N : C B :: C A : C r; or * C N × C r = C A × C B. But since C N × C r is equal to the fourth part C M H N, of the parallelogram G H, and C A × C B, the fourth part of the rectangle made by the axes; it follows, that the rectangle made by the two axes, is equal to the parallelogram made by two conjugate diameters.

*art. 293. *art. 30. *art. 86. *art. 72.

THEOREM VIII.

312. *The sum of the squares of any two semi-conjugate diameters* C M, C N, *will be always equal to the sum of the squares of the two semi-axes* C A, C B. *I say*, $\overline{CM}^2 + \overline{CN}^2 = \overline{CA}^2 + \overline{CB}^2$. Because of the right angled triangles C P M, C Q N: we have $\overline{CP}^2 + \overline{PM}^2 = \overline{CM}^2$, and $\overline{CQ}^2 + \overline{QN}^2 = \overline{CN}^2$. Therefore $\overline{CM}^2 + \overline{CN}^2 = \overline{CP}^2 + \overline{CQ}^2 + \overline{PM}^2 + \overline{QN}^2$. But * $\overline{CQ}^2 = A\,Pa$, * $\overline{CA}^2 - \overline{CP}^2$, or $\overline{CA}^2 = \overline{CQ}^2 + \overline{CP}^2$; and by the * same argument $\overline{CB}^2 = \overline{PM}^2 + \overline{QN}^2$. Consequently, $\overline{CM}^2 + \overline{CN}^2 = \overline{CA}^2 + \overline{CB}^2$, by substitution of equals for equals.

*art. 309. *art. 46. *art. 309.

These are the most material properties of the ellipsis; it remains now to say something of their areas, and the solids described by the rotation about either axes.

THE-

THEOREM IX.

Fig. 13. 313. *If from the center* O *of the ellipsis, with half of the first axis* OA *as radius, a quadrant of a circle* ANC *be described, then any ordinate* PM *to that axis is to that ordinate produced to the circle always in a given ratio of half the lesser axis to half the greater. I say,* PM : PN :: OB : OC.

For the square of PN is, by the property of the circle, equal to the difference between the squares of OA and OP; and the square of PM, by the property of the ellipsis, to the difference between the squares of OA and OP, as the square of OB is to the square of OA or OC; therefore $\overline{PM}^2 : \overline{PN}^2 :: \overline{OB}^2 : \overline{OC}^2$. And consequently, PM : PN : OB : OC.

COR. I.

314. If the ordinate PN be supposed to move from the position of OC towards A, or from A towards OC, by a parallel motion, the spaces OBMP: OCNP, or AMP, ANP, described by the parts PM, PN, of this line, will always be in the constant ratio of OB to OC: For the lines PM, PN, being constantly in that ratio, the spaces described at the same time, can be in no other.

COR. II.

315, If OM and ON are drawn, the triangles OMP, ONP, having the same base OP, are * *art.* 65.* as their altitudes PM, PN, or as OB, OC, as has been proved above; these triangles are therefore in the same ratio as the segments AMP, ANP. Consequently, the sectors OMA, ONA, the sums of these proportionals, are likewise in the same ratio of OB to OC.

COR.

COR. III.

316. Because the sector OAN, of the circle, is* $= \frac{OC \times AN}{2}$; we have $OC : OB :: \frac{OC \times AN}{2} :$ *art 175

$\frac{OB \times AN}{2} =$ to the sector OAM. Therefore the elliptical quadrant OBA will be equal to $\frac{OB \times ANC}{2}$. Consequently, the area of the whole ellipsis is equal to half the rectangle of half the lesser axis, and the circumference, whose radius is half the greater.

THEOREM X.

317. *The spheroid described by the rotation of the ellipsis about the greater axis, is to the sphere of the circle described upon that axis as a diameter, as the square of half the lesser axis is to the square of half the greater.*

For because PM : PN :: OB : OC, we have $\overline{PM}^2 : \overline{PN}^2 :: \overline{OB}^2 : \overline{OC}^2$ and as the areas of circles are as the squares * of their radii, the circle described about OA, by PM, is to the circle described by PN, as the square of OB is to the square of OC; and as this always happens, the solid described by OBMP is to the solid described by OCNP, as the square of OB is to the square of OC; and so is consequently also the semi-spheriod described by OBMA to the semi-sphere described by OCNA. * art. 170.

COR. I.

318. If the rectangle OD be made by the two semi-axes, and the diagonal OD drawn, then the

the ſemi-ſpheriod will be equal to the difference between the cylinder deſcribed by the rectangle O B D A, and the cone deſcribed by O A D. For the cylinder deſcribed by the rectangle O D, is to the cylinder deſcribed by the ſquare of O C, as the ſquares of the radii O B, O C, of their baſes; and the cone deſcribed by the triangle O A D is to the cone deſcribed by a triangle of the ſame axis O A, and whoſe radii of the baſe is O C, as the ſquare of O B is to the ſquare of O C; and therefore their differences between the greateſt cylinder and cone, is to the difference between the cylinder O D, and cone O A D, in the ſame ratio; but the ſemi-ſphere is equal to the difference between the circumſcribed cylinder and inſcribed cone. Conſequently, the ſemi-ſpheriod is likewiſe equal to the difference between the circumſcribed cylinder and inſcribed cone.

C O R. II.

319. Hence alſo, any part of the ſpheriod deſcribed by the ſegment O B M P, will be equal to the difference between the correſponding cylinder deſcribed by B P, and the cone deſcribed by O P L; for they are ſtill in the ſame ratio to the part of the ſphere deſcribed by O C N P, as their wholes; that is, as the ſquare of O B to the ſquare of O C.

Theſe ſolids and areas might have been inveſtigated by what has been ſaid in the ninth ſection of the firſt book.

SECT.

SECT. III.

Of the HYPERBOLA.

DEFINITIONS.

1. IF one end *f* of a ruler *f*D be fastened to a point, fix *f* in a plane, and a thread to another point, fix F in the same plane, whose length differs from that of the ruler, by less than the distance *f*, F between these points fixt; and the other end of the thread fastened to the other end D of the ruler; then the pin M, which serves to keep the thread tight, and the part M D quite close to the edge of the ruler, will describe by the motion of the ruler about the point *f*, the Hyperbola M A E. *Fig.* 14.

By fastening the end of the ruler to the point F, and the thread to the point *f*, and retaining the same length, you may describe in the same manner as before, the Opposite Hyperbola N a n.

2. The points fix F, *f*, are called the Foci.

3. The line A a, passing through the foci, and terminated by the opposite hyperbolas, the Axis.

4. The point C which bisects the axis, the Center.

5. If B b, perpendicular to the axis A a, be taken so that the distances A B, A b, are equal to that of the focus F from the center C, it is called the second axis.

The rest of the definitions of the hyperbola are the same as in the ellipsis.

COR.

COR. I.

320. It is evident that the distance A F, from the focus to the vertex of the axis is equal to the distance a *f*, of the other focus to the opposite vertex a; for when the point M falls on A, the part F M of the thread becomes equal to A F; and when the point m falls on a, the distance *f* m becomes *f* a; and since F M D is equal to *f* m d by definition, m d is equal to M D, when the edge of these rulers coincide with the axis A a; and therefore A F = a *f*.

COR. II.

321. Consequently the distance F *f* between the foci, is bisected by the center C; for C A = C a, and A F = *f* a; therefore C F = C *f*.

COR. III.

322. It is also evident, that the difference between the length of the ruler and that of the thread is equal to the first axis A a; for when the edge of the ruler *f* D coincides with that axis, then will *f* A − A F = *f* M − F M, or, because *f* A = *f* a + a A, a A = *f* M − F M. Consequently, if lines are drawn from the two foci *f*, F, to any point M in the hyperbola, the difference between these lines will be always equal to the first axis.

COR. IV.

323. Lastly, the square of C B, half the second axis, will be equal to the rectangle A F a, or to A *f* a; for because A B is equal to C F, by defin. a F will be equal to the sum of the sides C A and A B, and A F equal to the difference; therefore

* art. 46.* A F a = $\overline{CB}^2$.

THE-

THEOREM I.

324. *If from any point* M, *in the hyperbola, there be drawn a perpendicular* M P *to the first axis* a A *produced, and in that axis* C X *be taken so as* C A : C F :: C P : C X ; *then will* F M = A X, *and* fM = a X.

If half the sum of the lines fM, F M, be call'd D, then because in the triangle fM F the rectangle of the sum 2 D and difference 2 C A, of the sides fM, F M, is equal to the * rectangle of the sum and difference of the segments fP, and F P : Now their sum being fF, or 2 C F, and their difference 2 C P, we shall have 2 D × 2 C A = 2 C F × 2 C P ; or C A : C F :: C P : D ; whence C X = D equal to half the sum, to which adding, or from which subtracting half their difference C A, we shall have a X for the greatest fM and A X for the least F M. * *art.* 46.

N. B. When the perpendicular M P falls within the triangle, fF is the sum and C P half the difference of the segments fP, F P, and when that perpendicular falls without the triangle, fF is the difference and 2 C P the sum of the said segments.

THEOREM II.

325. *The square of any perpendicular* M P *to the first axis, is to the rectangle of the corresponding parts of that axis, as the square of half the second axis is to the square of half the first. I say,* $\overline{PM}^2 : A\,Pa :: \overline{CB}^2 : \overline{CA}^2$.

Because C X or C A + F M : C F :: C P : C A ; subtracting antecedent from antecedent, and consequent from consequent, and comparing the terms of the second ratio to the differences F M—

FM—AP:CP::AF:CA; and comparing the terms of the laſt ratio to the ſums of the antecedents and that of the conſequents; I. FM—FP : aP :: AF : CA.

Now if in the firſt proportion we compare the terms of the ſecond ratio to the ſums of the antecedents and that of the conſequents, aP + FM : CP : aF : CA: Laſtly, comparing the terms of the laſt ratio, to the differences, between the antecedents and that of the conſequents, II. FM + FP : AP :: aF : CA.

If the terms of the laſt proportion are multiplied by the reſpective terms of the third, and we take the ſquare of PM for the rectangle of the * ſum and difference of the ſides FM, FP, of the right angled triangle FPM, we get $\overline{PM}^2 : APa : \overline{CB}^2 : \overline{CA}^2$, becauſe* $AFa = \overline{CB}^2$. This is exactly the ſame as in the ellipſis, as well as all the ſucceeding demonſtrations, a few ſigns excepted.

* *art.* 46.

* *art.* 323

COR. I.

326. Since the ſquare of any perpendicular PM drawn from any point of the hyperbola to the axis, is always to the rectangle of the correſponding parts aP, AP, of that axis, as the ſquare of half CB, the ſecond axis to the ſquare of CA half the firſt; it is evident that theſe perpendiculars on either ſide of the axis, equidiſtant from the center, are always equal. Conſequently, the firſt axis is alſo a diameter, whoſe ordinates MP are at right angles to it.

COR. II.

327. If the ordinate paſſes through the focus F ſuch as FL, then the proportion $\overline{PM}^2 : APa :: \overline{CB}^2 :$

$\overline{CB}^2 : \overline{CA}^2$ becomes $\overline{LF}^2 : \overline{CB}^2 :: \overline{CB}^2 : \overline{CA}^2$, becauſe * $\overline{CB}^2 = AFa$, or *LF : CB :: CB : CA * art. 323. Therefore the ordinate which paſſes through the * art. 82, focus is the * parameter to half the firſt axis. * def. 6.

COR. III.

328. Since * $\overline{CA}^2 : \overline{CB}^2 :: CA : FL$, we * art. 76. have $\overline{PM}^2 : APa :: FL : CA$, by equality of ratios. Conſequently, the ſquare of any ordinate PM, is to the rectangle of the correſponding parts a P, A P, of the axis, as the parameter F L is to half the firſt axis C A.

COR. IV.

329. If there be drawn another ordinate NQ to the firſt axis, then * will $\overline{QN}^2 : AQa :: FL : CA$, * art. 325. and as $\overline{PM}^2 : APa :: FL : CA$, we ſhall have $\overline{QN}^2 : AQa :: \overline{PM}^2 : APa$, by equality * of * art 53. ratios; or alternately, $\overline{QN}^2 : \overline{PM}^2 : AQa : APa$. Conſequently, the ſquares of any two ordinates to the firſt axis, are as the rectangles of the correſponding parts of this axis.

COR. V.

330. It is evident that the oppoſite hyperbolas M A E, N a n, are equal, ſo that if the one was laid over the other, they would agree in every point; for if C Q = C P, then will A P = a Q, and A Q = a P; therefore, A P a = A Q a, and $\overline{PM}^2 = \overline{QN}^2$; or, P M = Q N. But this will happen every where, let the ordinates P M, Q N, be drawn where it will, provided they are equidiſtant from the center, they will always be equal.

Consequently, the opposite hyperbolas are equal in all respects.

COR. VI.

Fig. 16. 331. If from any point M in the hyperbola, a line M R be drawn perpendicular to the second axis B b; then because the rectangle A P a is equal
* *art.* 46. to * the difference $\overline{CP}^2 - \overline{CA}^2$, of the squares of C P and C A, we have $\overline{PM}^2 : \overline{CP}^2 - \overline{CA}^2 :: \overline{CB}^2 : \overline{CA}^2$; or because C P = R M, and P M = C R, that proportion will become $\overline{CR}^2 : \overline{RM}^2 - \overline{CA}^2 :: \overline{CB}^2 : \overline{CA}^2$; and comparing the last ratio to that of the sums of the antecedents and of the consequents, $\overline{CB}^2 + \overline{CR}^2 : \overline{RM}^2 :: \overline{CB}^2 : \overline{CA}^2$, or inversely, $\overline{RM}^2 : \overline{CB}^2 + \overline{CR}^2 :: \overline{CA}^2 : \overline{CB}^2$. *Which shews that the square of any ordinate* R M *to the second axis, is to the sum of the squares of half that axis and the distance* C R *of this ordinate from the center, as the square of half the first axis is to the square of half the second.*

PROB. I.

Fig. 15. 332. *To draw a line* M D *which shall touch the hyperbola in a given point* M.

From the foci *f*, F, draw two lines to the given point M; in *f* M take M L equal to M F; then the line M D, perpendicular to F L, will be the tangent required.

From any other point m, in m D, draw lines to the foci and to the point L; then because *f* m — L m, or its equal *f* m — F m is always less than
* *art.* 23. * *f* L, or is * equal A a, and as the difference be-
* *art.* 312. tween the lines drawn from the foci to any point in the

the hyperbola is always equal to the firſt axis; it is evident that the point m is not in the hyperbola. Conſequently the line MD touches the hyperbola only in the point M.

COR. I.

333. It is manifeſt, that the angles fMD, FMD, made by the lines drawn from the foci to the point of contact M, are equal: For becauſe $ML = MF$ by ſuppoſition, the perpendicular MD to the baſe FL of the iſoſceles triangle FML, will * biſect the oppoſite angle M. * *art.* 18.

COR. II.

334. If the line CD be drawn, then becauſe Cf is equal to CF, and FD equal to DL, the line CD will be* parallel and equal to half the line fL, that is, * equal to CA; if the tangent interſects the firſt axis in T and meets the ſecond in t, then by the paralleliſm *$fM : CD$, or $CA :: fT : CT$; or ſubtracting the conſequents from their antecedents, and comparing the differences with the conſequents, $fM - CA : fC :: CA : CT$: And if AX be taken equal to FM, then fM will * be equal to aX, and $CA : CP :: CF : CX$; but $CX = fM - CA$: Conſequently, $CP : CA :: CA : CT$, by equality of ratios. *Which ſhews that the firſt ſemi-axis* CA, *is a mean proportional between the diſtances of the ordinate* PM *drawn from the point of contact and the point* T, *where the tangent meets that axis, from the center* C. * *art.* 67. * *art.* 322. * *art.* 67. * *art.* 324.

COR. III.

335. If the line Kk, drawn perpendicular to the tangent at M, meets the firſt axis in K, and the ſecond in k, then becauſe MK, LF, are parallel, we have *$(fL)^2 CA : (fF)^2 CF :: fM : fK$; * *art.* 67.

fK; or, subtracting the terms of the first ratio from those of the second, $CA : CF :: fM - CA : CK$; but * $CA : CF :: CP : CX = fM - CA$: Therefore multiplying the terms of the last proporion by the respective ones of this, we get $\overline{CA}^2 : \overline{CF}^2 :: CP : CK$.

* art. 324.

COR.

336. Hence, since $\overline{CA}^2 : \overline{CF}_2 :: CP : CK$; we get by subtraction, $\overline{CA}^2 : \overline{CF}^2 - \overline{CA}^2$, or * $\overline{CB}^2 :: CP : PK$; and because * $PK : PM ::$ pM or $CP : pk$; we have $\overline{CA}^2 : \overline{CB}^2 :: pk : PM$, or Cp, by equality of ratios.

* art. 323. * art. 69.

COR.

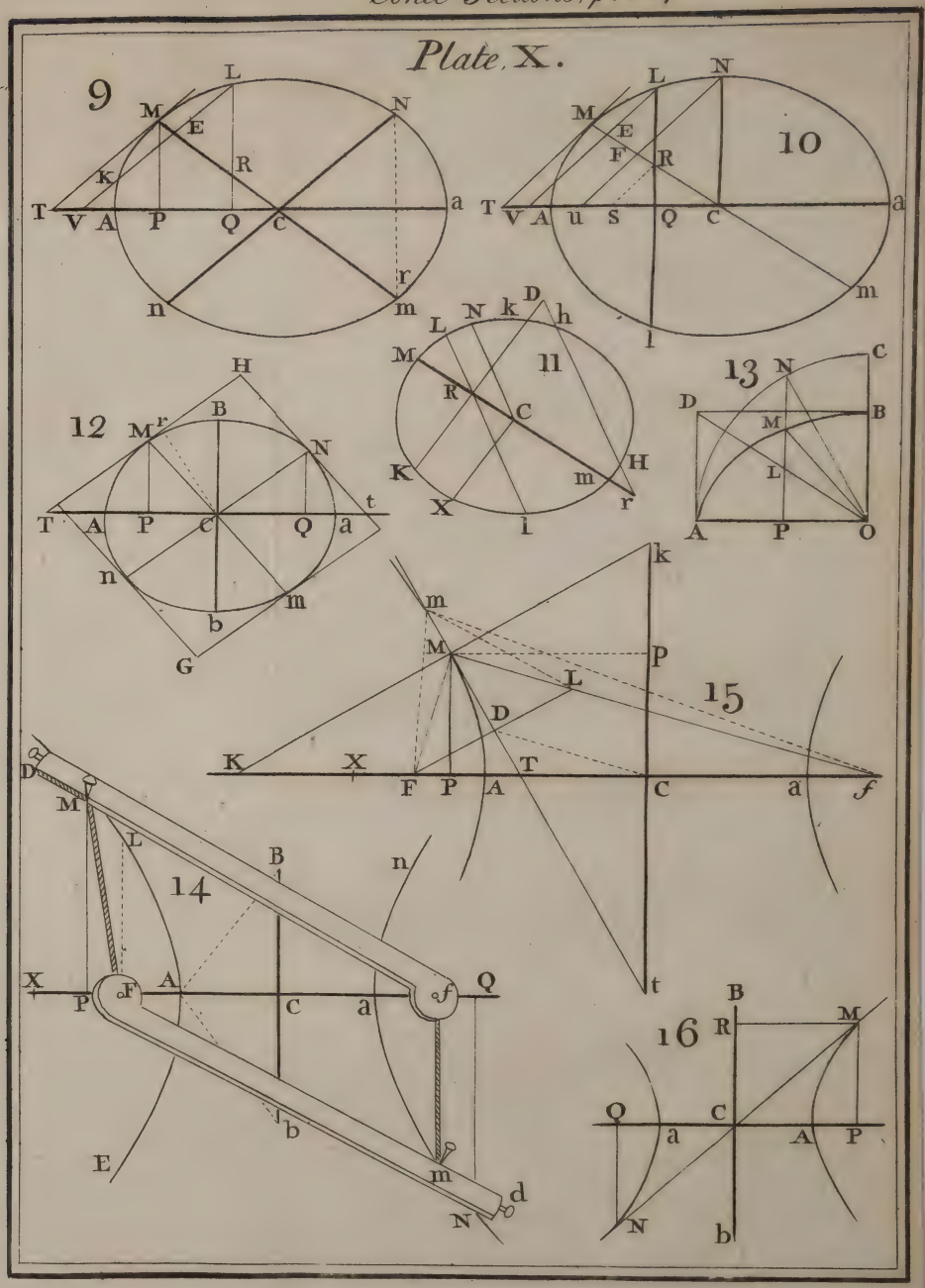
337. Because $CP \times CT : PK \times CT :: CP : PK$, we have $\overline{CA}^2 : \overline{CB}^2 :: CP \times CT : PK \times CT$; and as * $CP \times CT = \overline{CA}^2$; we have also $\overline{CB}^2 = PK \times CT$: And therefore the similar triangles CTt, MPK, give $PM : PK :: CT : Ct$; and * PM or $Cp : CB :: CB : Ct$, because $PK : CB :: CB : CT$. Which shews that the properties of the tangents are the same in respect to both axes.

* art. 334. * art. 85.

N. B. We have hitherto followed the same steps, in the demonstrations of the several theorems relating to the hyperbola, as in the ellipsis; and what has been said in regard to one, may equally be applied to both. We might in the same manner proceed, were it not necessary to mention some lines peculiar to the hyperbola, and as the rest of the properties of the hyperbola, are

more

Plate X.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more eaſily deduced from theſe lines than by the preceding method, we choſe to follow this latter.

DEFINITION.

The two lines n T, R S, drawn through the center C of the oppoſite hyperbolas, parallel to the lines A b, A B, which join the extremities of the two axes A a, B b, are called Aſymptotes. *Plate* XI. *Fig.* 17.

COR.

338. Hence, the tangent E F, drawn through either extremity of the firſt axis, and terminated by the aſymptotes, is equal to the ſecond axis B b, and biſected by the point of contact A: For ſince b C, A E, are parallel, as likewiſe b A and C E, the figure b E is a parallelogram; and therefore the oppoſite ſides b C, A E, are equal; by the ſame reaſon the oppoſite ſides C B, F A, of the parallelogram F B, are equal. Conſequently, b B = E F.

THEOREM III.

339. *If a line* R T, *drawn through any point* M *in the hyperbola, parallel to either axes, be terminated by the aſymptotes, the rectangle of its parts will be equal to the ſquare of the ſemi-axes to which it is parallel.*

I. For by reaſon of the parallels A E and P T, we have* $\overline{CA}^2 : \overline{AE}^2$ or $\overline{CB}^2 :: \overline{CP}^2 : \overline{PT}^2$; and by the property of the hyperbola, in reſpect to the firſt axis, we have * $\overline{CA}^2 : \overline{CB}^2 :: \overline{CP}^2 - \overline{CA}^2 : \overline{PM}^2$; ſubtracting the terms of the laſt ratio in the laſt proportion, from the reſpective ones of the firſt, we get $\overline{CA}^2 : \overline{CB}^2 :: \overline{CA}^2 : \overline{PT}^2 - \overline{PM}^2$. * *art.* 67. * *art.* 325.

* *art.* 46. Consequently, $\overline{CB}^2 = RMT$; because * $\overline{PT}^2 - \overline{PM}^2 = RMT$.

II. Because AC is parallel to pr, we have $\overline{CA}^2 : \overline{AF}^2$ or $\overline{CB}^2 :: \overline{pr}^2 : \overline{Cp}^2$; and by the property of the hyperbola, in respect to the second axis,

* *art.* 331. we have * $\overline{CA}^2 : \overline{CB}^2 :: \overline{pM}^2 : \overline{CB}^2 + \overline{Cp}^2$; and subtracting the terms of the last ratio in the first, from those of the last in the last, we get $\overline{CA}^2 : \overline{CB}^2 :: \overline{pM}^2 - \overline{pr}^2 : \overline{CB}^2$. Therefore $\overline{CA}^2 = rMt$; be-

* *art.* 46. cause * $\overline{pM}^2 - \overline{pr}^2 = rMt$.

COR. I.

340. Hence, if there be drawn any where two parallels *a*d, Ll, the rectangles of their parts terminated by the asymptotes and the hyperbola, will be equal: Through the points M, N, where these lines intersect the hyperbola, draw RT, Sn, parallel to one of the axis bB; then, by reason of

* *art.* 69. the parallelism, we have * MR : Ml :: NS : N*a*, and MT : ML :: Nn : Nd; multiplying the terms of one proportion by the respective terms of the other, we get RMT : LMl :: SNn : *a*Nd;

* *art.* 339. but $RMT = * \overline{CB}^2 = SNn$. Therefore $LMl = a\text{N}d$.

COR. II.

341. It follows also, that the parts ML, ul, of the same line drawn any how, terminated by the asymptotes and the hyperbola, are equal. For

* *art.* 340. because * $LMl = luL$; or $LM \times \overline{Mu} + ul = lu \times \overline{uM} + \overline{ML}$; by subtracting the same rect-

angle

angle LM×ul, from both, we get LM×Mu= lux uM; or LM = ul, by diviſion.

COR. III.

342. It is manifeſt, that any tangent terminated by the aſymptotes, is biſected by the point of contact; for ſince the parts Rm, MT, of any line RT, terminated by the aſymptotes and the hyperbola, are always equal, when that line becomes the tangent EF, the points M, m, coincide with the point of contact A. Conſequently, AF = AE, and the rectangle TMR or TmR, is always equal to the ſquare of half that tangent AE or AF. *Fig.* 18.

COR. IV.

343. It is no leſs evident, that the line Mm, parallel to the tangent EF, and terminated by the hyperbola, is biſected by the line CP, drawn through the center C and the point of contact A; for ſince AF = AE, PT will be * = PR; and * *art.* 67. as Rm = MT, it is evident that PM = Pm. Conſequently, aA is a diameter, and Mm its ordinate. *Which ſhews that all lines paſſing through the center, and meet the hyperbolas, are diameters.*

DEFINITION.

If from the extremity A of a diameter, two lines AB, Ab, are drawn parallel to the aſymptotes, ſo as meet a line Bb, parallel to the tangent at A; this line Bb, is ſaid to be the conjugate diameter to the diameter Aa.

COR.

344. It follows from this definition, that the conjugate diameter Bb, is equal to the tangent

EF to which it is parallel; since the opposite sides bC, AE and FA, CB, of the parallelograms bE, FB, are equal.

THEOREM IV.

345. *The square of an ordinate* PM *to any diameter* Aa, *is to the rectangle of the corresponding parts of that diameter, as the square of half its conjugate is to the square of half that diameter.*

I say, $\overline{PM}^2 : APa :: \overline{CB}^2 : \overline{CA}^2$.

*art. 339. * *art. 46.* Because * TMR, or * $\overline{PT}^2 - \overline{PM}^2 = \overline{AE}^2$; we have $\overline{PT}^2 - \overline{AE}^2 = \overline{PM}^2$; and the similar triangles CAE, CPT, give $\overline{PT}^2 : \overline{AE}^2 :: \overline{CP}^2 :$ * *art. 75.* $\overline{CA}^2$; or, * $\overline{PT}^2 - \overline{AE}^2 : \overline{CP}^2 - \overline{CA}^2 :: \overline{AE}^2 : \overline{CA}^2$. But $\overline{PT}^2 - \overline{AE}^2 = \overline{PM}^2$; $\overline{CP}^2 - \overline{CA}^2 = APa$, and $\overline{AE}^2 = \overline{CB}^2$. Consequently, $\overline{PM}^2 : APa :: \overline{CB}^2 : \overline{CA}^2$.

COR.

346. Since the square of an ordinate PM, to any diameter, is always to the rectangle of the corresponding parts of the diameter, as the square of half its conjugate is to the square of half that diameter; whether the diameter be an axis or not; it is evident, that whatever has been deduced from that property, in regard to the axes, will also be true in regard to any diameters.

THEOREM V.

347. *If a line* mM *intersects any diameter* Nn, *either within or without the hyperbola, the rectangle of its parts terminated by the hyperbola and this diameter, will be to the rectangle of the corresponding parts of that diameter, as the square of half the diameter*

diameter parallel to that line is to the ſquare of half the interſecting diameter. I ſay, m S M : N S n :: $\overline{CB}^2$: $\overline{CN}^2$.

By reaſon of the parallels L Q, T P, we have C Q : C P or C N : C S :: Q L : P T :: Q N : P S; and therefore the ſquares of theſe lines are alſo proportional, as likewiſe the ſquares of the terms C N, C S, are proportional to the differences between the ſquares of Q L, Q N, and P T, P S; but the difference between the ſquares of Q L, Q N, is equal to the ſquare of *C B; whence the ſquare of C N is to the ſquare of C S, as the ſquare of C B is to the difference between the ſquares of P T and P S: And if we ſubtract the antecedents from their conſequents, the ſquare of C N is to the difference between the ſquares of C S and C N, or the rectangle N S n, as the ſquare of C B is to $\overline{PT}^2 - \overline{CB}^2 - \overline{PS}^2$; but the difference between the ſquares of P T and C B * is equal to the ſquare of P m. Conſequently, $\overline{CN}^2$: N S n :: $\overline{CB}^2$: M S m; becauſe *$\overline{Pm}^2 - \overline{PS}^2$ = m S M, or alternately and by inverſion, m S M : N S n :: $\overline{CB}^2$: $\overline{CN}^2$.

* *art.* 342.

* *art.* 339.

* *art.* 46.

COR. I.

348. Hence, if any other line K k, parallel to the ſemi-diameter C X, meets the diameter n N, in the ſame point S, as M m, and the hyperbola in K, k; then will ($\overline{CN}^2$: N S n ::) $\overline{CB}^2$: M S m :: $\overline{CX}^2$: k S K; or alternately, $\overline{CB}^2$: $\overline{CX}^2$ :: M S m : k S K.

COR.

COR. II.

349. If the line H h, parallel to M m, meets k K, produced if neceſſary, in D; it is evident that $(\overline{CB}^2 : \overline{CX}^2 ::)$ MSm : kSK : HDh : KDk.

In general, the rectangles of the parts of two parallel lines, terminated by the hyperbola and any other line, are always as the rectangles of the correſponding parts of that line, terminated by the hyperbola and theſe parallels; and if one or two of theſe lines become tangents, the rectangles of their parts will become the ſquares of thoſe lines. This is a natural conſequent from the foregoing corollaries.

THEOREM VI.

Fig. 17. 350. *If from any point* N *in the hyperbola two lines* N Q, N V, *are drawn parallel to the aſymptotes, and being terminated by the ſame; their rectangle will be always the ſame, let that point be taken where it will.*

Let A B interſect the aſymptote in H, then both triangles Q N S, N V n, will be ſimilar to the triangles C H B, as having all their ſides parallel; and therefore N S : N Q :: C B : C H, and N n : N V :: C B : H B; multiplying the terms of one proportion by the reſpective terms of the other, we get S N n : N Q × N V :: $\overline{CB}^2$: C H × H B; but * S N n = $\overline{CB}^2$. Conſequently N Q × N V = C H × H B.

* art. 339.

COR.

351. Since the oppoſite ſides of the figure V Q are parallel, it is a parallelogram, and ſo the oppoſite ſides V N, and C Q are equal; and becauſe C H

CH is parallel to bA, and bC = CB, AH is also equal to HB; and therefore CQ × QN = CH × HA. Which shews, that if from any two points in one asymptote, two lines are drawn parallel to the other; and being terminated by the hyperbola, the rectangle of one part of the asymptote, and the corresponding line, will be equal to the rectangle of the other part and its corresponding line.

THEOREM VII.

352. *The parallelogram* GH, *made by any two conjugate diameters* Mm, Nn, *is equal to the rectangle made by the two axes* Aa, Bb. Fig. 19.

Let the lines AB, MN, joining the extremities of the axes and conjugate diameters, intersect the asymptote in L and R; then because CL × LA = CR × RM, we have CL : CR :: RM : LA, or CL : CR :: NM : AB; therefore the triangles BAC, NCM, which have their bases and altitude reciprocally proportional, are * equal, as well as their octuples, the parallelogram GH, and the rectangle made by the two axes. * art. 93.

THEOREM VIII.

353. *The difference between the squares of any two conjugate diameters* CM, CN, *is equal to the difference between the squares of the two axes* CA, CB.

Let C*b* be perpendicular to MN, intersecting AB in *a*; then the similar triangles CL*a*, CR*b*, give CL : CR :: L*a* : R*b*; and by the property of the asymptotes CL : CR :: NM : BA; therefore L*a* : R*b* :: NM : BA; or L*a* × AB = R*b* × NM. Hence, because the rectangle of the sum and difference between the segments

Mb,

M *b*, *b* N, of the triangle C N M, is equal to the rectangles of the sum and difference of the segments A *a*, *a* B, of the triangle C B A; their equals * the rectangles of the sum and differences of the sides C N, C M, and C B, C A, are also equal; and consequently, the difference between the squares of C M, C N, will be * equal to the difference between the squares C A and C B.

* *art.* 50.

* *art.* 46.

THEOREM IX.

Fig. 20. 354. *The spaces* B M Q S, F G R V, *terminated by two parallel lines* V Q, R S, *the asymptotes and the hyperbola, will be always equal.*

For since the parts B S, R G, of the same line R S, terminated by the asymptotes and the hyperbola are * always equal, it is evident, that if this line be supposed to move parallel to itself, the equal parts B S, R G, will describe equal spaces at the same time; and therefore when R S is moved to any situation V Q, the space B M Q S, described by B S, will be equal to the space F G R V, described by R G.

* *art.* 341.

COR. I.

355. Hence, if the lines B A, M P, are drawn parallel to the asymptote C V, and G D, F E, parallel to the asymptote C Q, the spaces A B M P, D G F E, terminated by these parallels, the asymptotes and the hyperbola are equal: For the triangles B A S, R D G, having the sides B S, R G, * equal, and all their sides parallel, are similar and * equal; and by the same argument the triangles M P Q, V E F, are equal. Therefore, by adding the equal triangles B A S, D R G, to, and subtracting the equal ones M P Q, V E F, from the equal spaces S B M Q, G R V F, we shall have

* *art.* 341.

* *art.* 25.

have the ſpace A B M P equal to the ſpace D G F E.

COR. II.

356. If from the points F, G, the lines F L, G K, are drawn parallel to the aſymptote C V, meeting the other C Q in L, K; then becauſe * C D × D G is equal to C F × E F, the rectangles C G, C F, are equal; by ſubtracting the rectangle C T which is common to both, the remainders L G, T E, are equal, and adding to theſe equals the ſame ſpace T G F, the ſums or ſpaces L K G F, D G F E, are equal; and ſince the ſpace D G F E has been proved * to be equal to the ſpace A B M P, the ſpaee L K G F is alſo equal to the ſpace A B M P. * art. 350. * art. 355.

COR. III.

357. If from the center C lines are drawn to the points B, M, G, F, the ſector C B M will be equal to the correſponding ſpace A B M P, and the ſector C G F, equal to the ſpace D G F E: For ſince * C A × A B = C P × P M, the triangles C A B, C P M, are * equal; from which taking the common ſpace C A H, and adding to the remainders the curvilinear ſpace H B M, the ſector C B M will be equal to the ſpace A B M P. * art. 350. * art. 93.

And by the ſame argument the ſector C G F is equal to the ſpace D G F E.

COR. IV.

358. Hence, the parts C L, C K, C A, C P, of the ſame aſymptote, terminated by the lines which include equal ſpaces, are proportional: For ſince * E F : D G :: V F : R G, and M P : A B :: M Q : B S; and becauſe the parts F V, M Q, are * art. 69.

* equal

* equal as well as R G, B S, we have E F : D G : : M P : A B, by equality of ratios; and as C L is equal to E F, and C K to D G, the laſt proportion becomes C L : C K : : M P : A B; but * M P : A B : C A : C P: Therefore C L : C K : : C A : C P, by equality of ratios.

*art. 350.

COR. V.

359. Whence, if ∴ C L : C K : C A, the ſpace K G F L will be equal to the ſpace K G B A, or the ſector C G F, equal to the ſector C B G: For if in the former proportion C A becomes C K, and C P, C A, the ſpace A B M P will become the ſpace K G B A, and ſtill remain equal to the ſpace K G F L; ſince the the line M F becomes F B in this caſe, and B G a tangent at G parallel to F B.

If ∴ C L : C K : C A : C P, it is evident from the preceding corollary, that the ſpaces L G, K B, A M, are equal; or the ſpace L F G K is to the ſpace L F M P as unity to 3: For ſince ∴ C L : C K : C A, the ſpace L F G K is equal to the ſpace K G B A; and as ∴ C K : C A : C P, the ſpace K G B A is equal to the ſpace A B M P; therefore theſe three ſpaces are equal.

It may be proved in general, that if there be any number *m* of continual geometrical proportionals, ſuch as C L, C K, C A, C P, *&c.* the lines drawn to the points L, K, A, P, parallel to the aſymptote C V, will include *m*—1 equal ſpaces; and therefore the firſt L G, will be to the ſum of them all, as unity is to the number *m*—1.

From this laſt corollary ariſes an eaſy and ſhort method for computing logarithms; and nothwithſtanding that ſome think they ſhould be deduced from the properties of numbers, we ſhall here explain their nature and the manner of computing

puting them; in this we follow Sir *Isaac Newton*, as may be seen in his Works translated by Mr. *Colson*; and the rather, on account that those who pretend to deduce them from the properties of numbers, are obliged to make use of so far-fetched principles as I, and I believe a great many, could never understand.

DEFINITION.

The Logarithms of numbers, which are in a geometrical progression, are such numbers as are in an arithmetical progression; or, if C L, C K, C A, C P, express any numbers, and C L being unity; the spaces L G, L B, L M, will express the logarithms of these numbers C K, C A, C P: And if C A expresses unity, the space A M expresses the logarithm of any number C P greater than unity, and A G, the logarithm of any number less than unity; and therefore the logarithm of unity is nothing.

The tables of logarithms contain the natural numbers from unity to 10,000 or to 100,000, and against them stand their logarithms; and by the nature of the logarithms the sum of the logarithms of any two numbers will be that of their product; the difference that of their quotient; and half, or one third of the logarithm of any number, will be that of its square or cube root; and lastly, twice or thrice the logarithm of any number, will be the square or cube of this number.

From whence it appears, that instead of multiplying numbers together, you only add their logarithms, and the sum is the logarithm of their product: To divide a number by another, you subtract the logarithm of the divisor from that of the

the dividend, and the difference is the logarithm of the quotient.

And inſtead of extracting any root of a number, you divide the logarithm by the exponent of the power; or, inſtead of ſquaring or cubing any numbers, you take the double or triple of its logarithm, and againſt the quotient or product the root or power is found.

PROB. II.

Fig. 20. 360. *To find the ſpace* A M, *or which is the ſame, to find the logarithm of any given number* C P, *greater than unity.*

Let G B M be an equilateral hyperbola, ſo that the aſymptotes C V, C P, may be at right angles to each other, and let C A expreſs unity; A B any number a; and A P, x; ſo that C P, or $1+x$, expreſſes the given number: Now as the rectangle of C A, and A B *(a)* is always *art.* 350 equal * to the rectangle of C P and P M, we have $a = 1+x \times PM$; or, dividing by $1+x$, $PM = \frac{a}{1+x}$: If this value of the perpendicular P M, be reduced into an infinite ſeries by a continual diviſion, we ſhall have $PM = a \times$ by $1 - x + xx - x^3 + x^4 -$ &c. and therefore the *art.* 232. area A B M P will be * $= a \times$ by $x - \frac{1}{2}xx + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 -$ &c. or this value will be the logarithm of the number $1+x$.

If the logarithm or are A B G K, of any number C K leſs than unity be required, by ſuppoſing A K $= x$; then by what has been ſaid $KG = \frac{a}{1-x}$ and

and by the same argument as before $a \times$ by $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 +$ &c. will express the area A G, or the logarithm of the number C K less than unity. It must be observed, that as the logarithm of unity is nothing, and that of any number greater than unity positive, that of any number less than unity must be negative; and since x cannot be greater than unity otherwise the logarithms above, become impossible: We must find an expression by which the several logarithms may be found, one after another.

COR. I.

361. If the logarithm of $1 - x$ be subtracted from the logarithm of $1 + x$, we shall have the logarithm of $\frac{1+x}{1-x}$, by the nature of logarithms, but as the logarithm of $1 - x$ is negative, instead of subtracting, it must be added, which gives $2a \times$ by $x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 +$ &c. for the logarithm of $\frac{1+x}{1-x}$; or, if $A = 2ax$, $B = x \times A$, $C = x \times B$, $D = x \times C$, &c. this expression will become $A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D +$, &c.

As the number a, or the rectangle A D may express any number, so there may be various sorts of logarithms, when $a = 1$, then the logarithms are called Hyperbolic, or natural.

It has been found convenient in practice, that the logarithm of 10 should be unity, and the common tables have been constructed accordingly; therefore the number expressed by a must be found, which may be done by finding the natural logarthm of 10 in the following manner.

Let n expreſs the logarithm of $\frac{5}{4}$, and m that of $\frac{1}{1}\frac{2}{2}\frac{8}{5}$; then I ſay that $10n+3m$ will be the natural logarithm of 10; for if x be the logarithm of 2, and y that of 5: then becauſe $\frac{5}{4}=\frac{5}{2\times 2}$; the logarithm y of 5, minus twice the logarithm x of 2, will be equal to the logarithm n of $\frac{5}{4}$; that is, $y-2x=n$; and ſince $\frac{128}{125}=\frac{2^7}{5^3}$, ſeven times the logarithm x of 2; minus three times the logarithm y of 5, will be equal to the logarithm m of $\frac{1}{1}\frac{2}{2}\frac{8}{5}$; that is, $7x-3y=m$; this will appear evident from what has been ſaid above, for the logarithm of any power of a number is equal to the logarithm of that number multiplied by the exponent of the power; and the logarithm of the quotient of one number divided by another, is equal to the difference between the logarithm of the diviſor and the dividend.

Now if $y-2x=n$ be multiplied by 3, it will become $3y-6x=3n$, which being added to $7x-3y=m$, gives $x=3n+m$; and ſince $y-2x=n$, gives $y=n+2x$, by tranſpoſition, and $x=3n+m$, multiplied by 2, gives $2x=6n+2m$; the ſum of this and $y=n+2x$, gives $y=7n+2m$; and becauſe 2×5 is 10, the ſum of the logarithms x and y, gives $x+y=10n+3m$, for the logarithm of 10.

EXAMPLE I.

362. *To find the logarithm of* $\frac{5}{4}$.

Suppoſe $\frac{1+x}{1-x}=\frac{5}{4}$; by multiplying croſſwiſe $4+4x=5-5x$, or $9x=1$ by tranſpoſition, hence

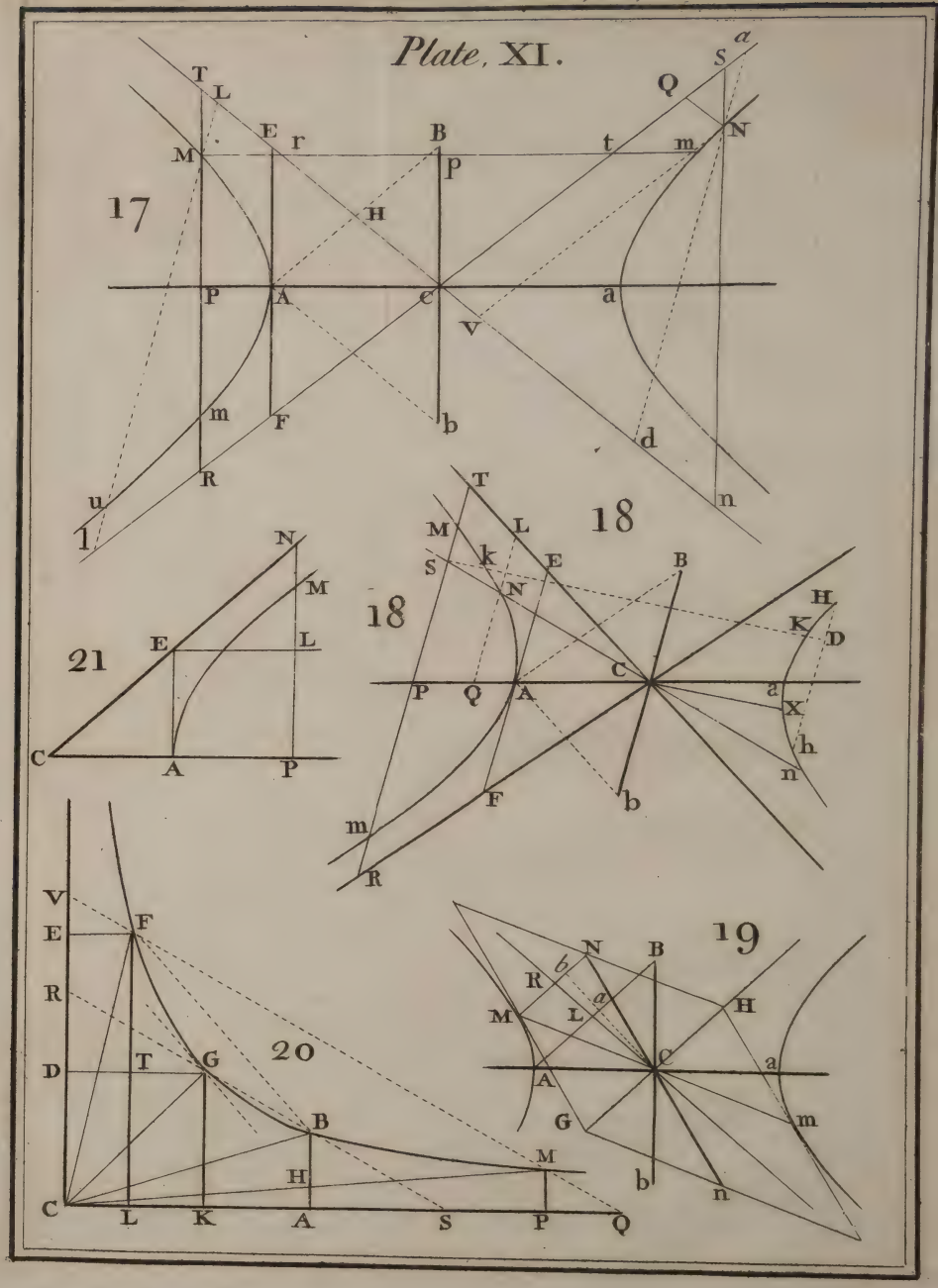
Plate. XI.
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hence $x = \frac{1}{9}$; by writing the value of x into those of A, B, C, &c. we get $A = \frac{2}{9}$, $B = \frac{A}{81}$, $C = \frac{B}{81}$, $D = \frac{C}{81}$, &c. which being reduced into decimals, gives

$$\begin{aligned} \cdot22222,22222,222222 &= A \\ 274,34842,249657 &= B \\ 3,38701,756168 &= C \\ 4181,503162 &= D \\ 51,623496 &= E \\ 637327 &= F \\ 7868 &= G \\ 97 &= H, \end{aligned}$$

And $A + \frac{1}{3}B + \frac{1}{5}C + \frac{1}{7}D +$, &c. gives

$$\begin{aligned} &\cdot22222,22222,222222 \\ &91,44947,416552 \\ &67740,351234 \\ &597,357595 \\ &5,735944 \\ &57938 \\ &606 \\ &6 \end{aligned}$$

$$\cdot22314,35513,142097 = n.$$

And $2\cdot23143,55131,42079 = 10n.$

EXAMPLE II.

363. *To find the logarithm of* $\frac{128}{125}$.

Suppose $\frac{128}{125} = \frac{1+x}{1-x}$; by reducing this equa-

tion we get $x = \frac{3}{253}$; and $A = \frac{6}{255}$, $B = \frac{9\,A}{253}$, $C = \frac{9\,B}{253^2}$, $D = \frac{2\,C}{253^2}$; these numbers being reduced into decimals give

.02371,54150,197628 = A
33345,113215 = B
4,688497 = C
659 = D

And .02371,54150,197628
11115,037738
937699
94

.02371,65266,173159 = m,

And .07114,95798,519477 = $3m$
2.23143,55131,420970 = $10n$

2.30258,50929,94045 = $10n + 3m$ = log.10.

As this number multiplied by a, is equal to unity, or to the tabular logarithm of 10, by dividing unity by this number we shall have a = 0.43429.44819,03252

Having thus found the value of a, the tabular logarithms are found as follows.

EXAMPLE III.

364. *To find the logarithm of* 9.

Suppose $\frac{1+x}{1-x} = \frac{10}{9}$, then will $x = \frac{1}{19}$, $A = \frac{2a}{91}$

B =

$B = \frac{A}{19^1}$, $C = \frac{B}{19^2}$, &c. and therefore these numbers reduced into decimals, give

.04571,52086,21395 = A
12,66349,26926 = B
3507,89271 = C
9,71715 = D
2691 = E

.04571,52086,21395
4,22116,42309
701,57856
1,38816
299

.04575,74905,60675 = log. $\frac{1}{9}^{\circ}$.

As the logarithm of 10 is unity, and 10 divided by $\frac{1}{9}^{\circ}$ gives 9, the logarithm of $\frac{1}{9}^{\circ}$, just now found, subtracted from unity, gives 0.95424,25094,39325 for the logarithm of 9.

And because the square root of 9 is 3, half the logarithm of 9 gives 0.47712,12547,19662, for the logarithm of 3.

EXAMPLE IV.

365. *To find the logarithm of* 8.

Suppose $\frac{1+x}{1-x} = \frac{81}{80}$; then will $x = \frac{1}{161}$, $A = \frac{2a}{161}$, $B = \frac{A}{161^2}$, $C = \frac{B}{161^2}$: And $A + \frac{1}{3}B + \frac{1}{5}C$, gives

.00539,49625,08115
693,76985
1604

.00539,50318,86704 = log. $\frac{11}{10}$. Now

Now since $9 \times 9 = 81$, and 81 divided by $\frac{81}{80}$ gives 80, the logarithm of $\frac{81}{80}$, subtracted from twice the logarithm of 9, found above, gives 1.90308,99869,91946, for the logarithm of 80; and as 80, divided by 10, gives 8, taking unity the logarithm of 10, from that of 80, we shall have 0.90308,99869,91946, for the logarithm of 8.

As $2 \times 2 \times 2 = 8$; one third of the logarithm of 8 gives 0.30102,99956,63982, for the logarithm of 2.

And because $2 \times 3 = 6$, by adding the logrithm of 2 and 3, we shall have 0.77815,12503,83644, for the logarithm of 6.

EXAMPLE V.

366. *To find the logarithm of* 7.

Suppose $\frac{1+x}{1-x} = \frac{49}{48}$; then will $x = \frac{1}{97}$, $A = \frac{2a}{97}$ $B = \frac{A}{97^2}$, $C = \frac{B}{97^2}$; and $A + \frac{1}{3}B + \frac{1}{5}C$, gives

.00895,45254,00067
3172,32628
20228

.00895,48426,52923 = log. $\frac{49}{48}$.

Now because $6 \times 8 = 48$, and 48 multiplied by $\frac{49}{48}$, gives 49; the logarithm of $\frac{49}{48}$ being added to the sum of the logarithms of 6 and 8 found above, gives 1.69019,60800,28513, for the logarithm of 49; or because $7 \times 7 = 49$, half this logarithm 0.84509,80400,14256 will be the logarithm of 7.

These examples are sufficient to shew the method of computing the logarithms of all the numbers;

bers; and it may be obſerved that the logarithms of large numbers are found by one ſtep, or two at moſt; or are found only by addition and ſubtraction: For becauſe the logarithms of even numbers, or which are the product of any number of factors, are found by adding thoſe of their factors; ſo that thoſe of the prime numbers are only to be found by the ſeries above; and this is eaſily done by having the logarithms next under and above them; for, if any three numbers exceed each other by unity, the ſquare of the mean will always exceed the product of the two extremes, by unity: Thus, to find the logarithm of 11, the next prime number; the ſquare of 11 is 121, and the product of 10 and 12 is 120; then finding the logarithm of $\frac{121}{120}$, to which adding the logarithm of 10×12, and half the ſum will be the logarithm of 11.

And to find the logarithm of 13, the ſquare of which is 169, and the product of 12 by 14 is 168; then the logarithm of $\frac{169}{168}$, added to the logarithm of 12×14; half this ſum will be the logarithm of 13; and in this manner the logarithm of any prime number is found.

It muſt be obſerved, that when the logarithm of any fraction is to be found, the value of x is always expreſſed by a fraction; whoſe numerator is the difference, and denominator the ſum of the numerator and denominator of the given fraction.

Thus, if $\frac{1+x}{1-x} = \frac{121}{120}$, then will $x = \frac{1}{241}$; and if $\frac{1+x}{1-x} = \frac{289}{288}$; then will $x = \frac{1}{577}$.

As the right uſe of the tables of logarithms is of great importance, we ſhall explain, in a few

words, the moſt remarkable difficulties: Firſt, when the logarithm of any number is to be found greater than any in the tables; when a logarithm is given, and the number anſwering to it, is not to be found exactly; or to find the logarithms of vulgar and decimal fractions; we ſhall explain thoſe four caſes one after another.

EXAMPLE I.

367. *To find the logarithm of any given number, not exceeding the number of places, to which the logarithms are computed.*

The leaſt tables are computed to 10.000; ſo if the logarithm of any number leſs than 10.000 is wanted, it may readily be found in the tables; thus the logarithm of 7987, will be found oppoſite to that number in the tables to be 3.9023837.

Before we find the logarithm of any number of above five places, it muſt be obſerved, that the index, or characteriſtic, of the logarithms of one place, or under 10, is 0; that of two places, or between 10 and 100, is 1; that of three places, 2; that of four, 3; and ſo on; that is, the index is always unity leſs than the number of places of the given number.

Now to find the logarithm of a number, as 56789: I divide it by 10; which gives 5678.9. Becauſe the logarithm of 5678 is .7541954, omitting the index; and that of 5679, is .7542719; as the firſt is leſs and the ſecond greater, than the logarithm wanted, take their difference, which is 765; take likewiſe the difference between the leaſt number 5678, and the given number 5678.9, which is .9; multiply theſe differences together, and add the product 688.5, or only 688, to the

logarithm

logarithm of 5678, and the ſum .7542642, will be the logarithm ſought, by prefixing the index 4 to it.

The ſame logarithm will be found by multiplying the difference 765, by the difference .1, between the given number and the greater, and ſubtracting the product 76.5, from the logarithm of the greater. Hence follows this

GENERAL RULE.

To find the logarithm of any number greater than 10.000.

Strike off the figures above four, as decimals 5678.9. take the difference 765, between the logarithms of the numbers next above and below the given one.

Multiply this difference by the difference .9 or .1, between the given number and that next above or below it.

Add the product to the leſſer, or ſubtract it from the greater, then the ſum or difference will be the logarithm ſought.

EXAMPLE II.

368. *To find the number of a given logarithm.*

If the logarithm is found exactly in the tables, the number will be oppoſite to it; but if it cannot be found, obſerve the following

GENERAL RULE.

Take the difference between the logarithms next above and below the given one.

Take likewiſe the difference between the given logarithm and one of the others.

Divide this laſt difference by the former; and add or ſubtract the quotient to the former number encreaſed by as many cyphers as the quotient contains

tains places, according as the logarithm taken is leſs or greater than the given logarithm.

Let the given logarithm be - - 3.7890423
Then the logarithm of 6152 is leſs - 7890163
And that of 6153 is greater - - 7890869
The firſt difference is - - - - - - 706
And the ſecond - - - - - - - 260
Therefore the number ſought is - 6152.367

EXAMPLE III.

369. *To find the logarithm of any vulgar fraction.*

Subtract the logarithm of the denominator from that of the numerator, and the difference will be the logarithm wanted.

Thus the logarithm of $\frac{7}{5}$ is equal to the difference .1461280, between the logarithms .8450980 of 7, and that .6989700 of 5.

But if the fraction is proper, that is, if the numerator is leſs than the denominator, the numerator muſt be multiplyed by 10, or 100, ſo as to become greater than the denominator; then ſubtract from the index as many units as there has been cyphers added; thus the logarithm of $\frac{3}{4}$ by multiplying by 10, to get $\frac{30}{4}$, is .8750613; and ſubtracting unity, —1.8750613; that of $\frac{1}{80}$, or $\frac{100}{80}$ is 0.5740313; and ſubtracting 2 gives —2.5740313 for the logarithm required.

The logarithm of $\frac{2}{345}$, or of $\frac{2000}{345}$ is 0.7632109, and ſubtracting 3, —3.7632109.

EXAMPLE IV.

370. *To find the logarithm of any decimal fraction.*

Conſider decimals as whole numbers, and ſubtract

tract as many units from the index as there are decimal figures.

Thus, the logarithm of .125, considered as whole numbers, is 2.0969100; and subtracting 3 from the index gives —1.0969100.

The logarithm of .00126, by considering 126 as whole numbers, is 2.1003750; and subtracting 5 from the index gives —3.1003750.

The logarithm of any mixt number is also found in the same manner, by considering them as whole numbers, and subtracting as many units from the index as there were decimals; or prefixing the index of the whole number as if there were no decimals. Thus, the logarithm of 2.134, considered as whole numbers, is 3.3291944; and subtracting 3 from the index gives .3291944; this appears from the numbers also, since there is but one figure, namely 2, of whole numbers; the index must by course be nothing.

Having given the manner of constructing the logarithms, as likewise how to find them in the tables, so as to answer any given numbers, the application of them will be shewn in the next section; so there remains now to determine the solid contents of hyperboloids.

PROB. III.

371. *To find the content of the solid described by the segment* A M P, *about the first axis* C P. Fig. 21.

Draw the asymptote C N, the tangent A E, and E L, parallel to the axis intersecting the ordinate P M in L; and let that ordinate produced meet the asymtote in N.

Then because A E or P L is equal to half the second axis, and * $\overline{AE}^2$ or $\overline{PL}^2 = \overline{PN}^2 - \overline{PM}^2$ * art 339.

or,

*art. 170. or, by tranſpoſition, $\overline{PM}^2 = \overline{PN}^2 - \overline{PL}^2$; and ſince the areas of circles are * as the ſquares of their radii, the difference between the circles deſcribed by PN and PL, or the area deſcribed by LN, in the rotation of the figure, will be equal to the area deſcribed by the ordinate PM; and as this always happens, the ſolids deſcribed by the triangle ELN, will be equal to the hyperboloid deſcribed by the correſponding ſegment APM.

Which ſhews, that the hyperboloid deſcribed by any ſegment AMP, is equal to the difference between the fruſtrum of the cone, deſcribed by the ſpace AENP, and the cylinder deſcribed by the rectangle AELP.

BOOK.

BOOK III.

Of PLANE TRIGONOMETRY.

SECT. I.

DEFINITIONS.

1. TRIGONOMETRY is the ſcience which conſiders the ſides and angles of triangles, and is divided into plane and ſpheric; the plane conſiders right lined triangles; and the ſpheric, thoſe which are formed on the ſphere by its great circles. It is only the former which we propoſe to treat of, the other being of no uſe to military gentlemen, for whom this Work is chiefly intended.

2. In order to ſolve the different problems in trigonometry, it is neceſſary not only that the circumference of a circle be divided into degrees and minutes, as has been ſhewn in the ſixth ſection of the firſt book, but likewiſe certain lines within and without the circle are to be expreſſed by parts of the radius.

3. The ſine of an arc is the line drawn from one of its extremities, perpendicular to the diameter which paſſes through the other: Thus if PM be perpendicular to the diameter A a, this line PM is the ſine of the arc AM; and becauſe the arc Ma is likewiſe determined by the ſame line PM, it is alſo *Plate* XII. *Fig.* 1.

alſo the ſine of the ſupplement M a to the arc A M.

4. The coſine of an arc A M is the part C P of the diameter terminated by the center C and the ſine P M of that arc.

If the radius C B be drawn at right angles to the diameter A a, and M Q perpendicular to it; then becauſe C P and Q M are equal, and Q M being the ſine of the arc M B; the coſine C P of an arc A M, is equal to the ſine of its complement M B.

5. The verſed ſine of an arc A M, is the part A P, of the diameter terminated by its ſine and that arc.

6. The tangent of an arc A M, is the line A T, which touches it at one extremity A, and is terminated by a line drawn from the center C thro' the other M.

7. The ſecant is the line C T drawn through the center, and terminated by the tangent.

8. Laſtly, the cotangent and coſecant of an arc A M, are the tangent B t, and ſecant C t, of its complement B M.

There are tables which contain the ſines, tangents, and ſecants of every arc, from a minute to 90 degrees, expreſſed by parts of the radius divided into 10000000 equal parts; the principles on which their computation depends are contained in the following propoſitions.

PROB. I.

Fig. 1. 372. *Any arc* A M *being given or expreſſed by parts of the radius, to find its ſine* P M.

If the arc A M be called z, and its ſine P M, * *art.* 252. x; then becauſe * $z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 +$, &c. by cubing each ſide we get $z^3 = x^3 + \frac{1}{2}x^5$, neglecting the

the higher powers, and $z^5 = x^5$; the first divided by 6 gives $\frac{1}{6}z^3 = \frac{1}{6}x^3 + \frac{1}{120}x^5$; and the second by 120, $\frac{1}{120}z^5 = \frac{1}{120}x^5$. These values of x and its powers being substituted into the value of the arc, gives $x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5$, nearly.

The higher powers of x and z have been neglected, because the expression of the sine is sufficiently exact, as will appear hereafter: For, when the arc is but small, the difference between the arc and the sixth part of its cube, will give the sine of that arc nearly.

PROB. II.

373. *Any arc* A M *being given or expressed by parts of the radius, to find the cosine* C P *of that arc.*

Let the cosine C P be called v; then because of the right angled triangle C P M, the difference between the squares of C M (1) and P M (x) will * be equal to the square of the cosine; that is, we have $vv = 1 - xx$, or $v = \sqrt{1 - xx}$; and the square root of $1 - xx$, by the lemma in art. 249, will be $v = 1 - \frac{1}{2}xx - \frac{1}{8}x^4 - \&c.$ Now because we have * $x = z - \frac{1}{6}z^3 +$, &c. the square of which is $xx = zz - \frac{1}{4}z^4$, and $x^4 = z^4 -$, &c. by neglecting the higher powers. These values of xx and x^4, being substituted into $v = 1 - \frac{1}{2}xx - \frac{1}{8}x^4$, give $v = 1 - \frac{1}{2}zz + \frac{1}{24}z^4$ nearly for the cosine C P required.

* art. 49.

* art. 372.

It is evident that the versed sine A P is equal to (C A — C P) $\frac{1}{2}zz - \frac{1}{24}z^4$ nearly.

COR.

COR.

374. If the tangent A T be called t; then because the cosine C P is to the sine P M, as the radius C A is to the tangent A T; it is evident, that the sine P M $(z-\frac{1}{6}z^3+\frac{1}{120}z^5)$ divided by the cosine C P $(1-\frac{1}{2}zz+\frac{1}{24}z^4)$ gives $t=z+\frac{1}{3}z^3+\frac{2}{15}z^5$ nearly.

PROB. III.

375. *Any arc* A M *being given in degrees, to express that arc in parts of the radius.*

The circumference is to the diameter, or the semi-circumference to the radius, as unity is * to 3.14159,295389: If, therefore, we say as the semi-circumference, or 180, is to the number of degrees of the arc, so is the number 3.14159, to that arc expressed in parts of the radius. Wherefore, if the arc be one minute, then as 180×60, or 10800 minutes is to one minute, so is 3.14159, &c. to that arc 0.00029,08882,088 expressed in parts of the radius.

*art. 252.

By means of the three last problems, and the following theorem, the sines, tangents, and secants, may be computed, as will appear by the following

EXAMPLE.

376. *To find the sine, cosine, and tangent of an arc of one minute.*

If the value of the arc z, found in the last problem, be substituted into * $x=z-$ &c. we shall find 0.00029,08882,086 for the sine; if the square of that value be substituted into * $v=1-\frac{1}{2}zz$, we shall find 0.99999,99576,92 for the cosine; and lastly, $t=z$, gives 0.00029,08882,086 for the tangent required, true to the last figure.

*art. 372.

*art. 373.

THE

THEOREM I.

377. *If there be taken three arcs* AN, AM, AL, *in an arithmetical progreſſion, the radius being unity; the ſum of the ſines of the extremes will be equal to twice the rectangle made by the ſine of the mean and the coſine of half their difference; and the ſum of their coſines to twice the rectangle made by the coſine of the mean and that of half their difference.* Fig. 2.

Draw the chord LN, to the difference between the leaſt AN and the greateſt AL; let the radius CM biſect at right angles the chord NL in R; then if the ſines LD, MP, NQ, of theſe arcs are drawn, RI parallel to them, and ER, FN, parallel to the radius CA; the ſimilar triangles CPM, CIR, gives * CM : CP :: CR : CI, and CM : PM :: CR : IR. Therefore CI = CP × CR, IR = PM × CR, becauſe the radius CM is unity. * art. 67.

Now becauſe LR, RN, are equal, and ER, FN, parallel; LE and EF are alſo * equal; ſo then DL, IR, and QN, are in an arithmetical pregreſſion; that is, the ſum of DL, QN, the extremes is equal to twice the mean IR. Therefore DL + QN = 2 PM × CR. * art. 67.

And ſince LE and EF are equal, ER or DI is half of FN or DQ; that is, DI and IQ are equal; and ſo CD + CQ (= 2 CI) = 2 CP × CR.

COR. I.

378. If the arc MN or ML be one minute, its coſine CR will be * 0.99999,99576,920 : Therefore LD + NQ = PM × 1.99999,99153,84, and CD + CQ = CP × 1.99999,99153,84, * art. 376.

By means of theſe two laſt equations, the ſines

and cosines of any arc from a minute to 30 degrees, may be found by a single multiplication.

EXAMPLE.

379. Let LD be the radius; then will PM be the sine of 89°:59′; that is, PM will be
* art. 376. * .99999,99576,92; and the cosine CP, 0.00029,08882,086; these values being substituted into the former equations, give NQ= .99999,98665,067 for the sine, and CQ= .00058,18760,36 for the cosine of the arc 89°:58′.

The next sine and cosine is found by supposing DL to be the sine of 89°:59′, and PM that of 89°:58′; and the operations may be thus continued to the finding of sines and cosines of arcs from a minute to 30 degrees.

COR. II.

380. If the arc AM be 30 degrees, its sine PM will be equal to half the radius; or, because the radius is unity, that sine will be $\frac{1}{2}$: Therefore
* art. 377. the equation * LD+NQ=2PM×CR, will become LD+NQ=CR in this case. *Which shews that the sum of the sines of two equidistant arcs from 30 degrees, is always equal to the sine of half their difference.*

And because of the similar triangles CPM, LFN, we have CM(1):PM($\frac{1}{2}$)::LN:FN or ::2LR:DQ: Therefore LR=(DQ)CQ—CD. Consequently, *the difference between the cosines of any two equidistant arcs from 30 degrees, is always equal to the sine of half their difference.*

By means of these equations the sines and cosines of any arc between 30 and 60 degrees, may be found by subtraction only, when the sines and cosines of all the arcs from a minnte to 30 degrees are known.

The

The tangents may be found much in the same manner as the sines and cosines, by the expression given for that purpose; or because the cosine C P is to the sine P M, as the radius C A is to the tangent; the sine and cosine of an arc being given, the tangent is easily found by this last proportion. *Fig* 1.

The cosine C P is also to the radius C M as the radius C A to the secant C T. Consequently, the cosine of an arc being given, the secant may be found.

Explanation *of the* Tables.

At the top of the page is placed the number of degrees; in the first vertical column to the left, the minutes; in the second vertical column the sines; in the third the tangents; in the fourth the secants; and then follow the logarithmical sines and tangents; and in the opposite page are the sines, tangents, and secants of their complements, in the same order as in the first.

Thus, if the sine, tangent, and cosine of 39°: $15'$ is required, look for 39 degrees at the top of the page, and for the 15 minutes in the first vertical column to the left; then the number 63270.53 against 15, and under the word sine, will be the sine; the number 81703,43, against 15, and under the word tangent is the tangent; and in the opposite page the number 77439,26, in the same horizontal line under the word sine, is the cosine of the arc required.

Again, the sine of 34 degrees 40 minutes, is 56880,11, the tangent 69157,24, the secant 121584,23, and cosine 82247,51.

If the sine or cosine of any angle greater than 90 degrees is required, subtract from 180 the given number, and the sine or cosine of the difference will be that required: For it has been

ſhewn, that the ſine or coſine of any arc under 90 degrees, was alſo the ſine or coſine of its ſupplement. Thus, if the ſine of 100 degrees is wanted, ſubtract 100 from 180, and the ſine 98480,77 of the difference will be that required.

As the tables are computed to minutes only, and it is ſometimes required to find them to a greater accuracy, we ſhall ſhew how to find them nearer by approximation, or from the given ſine or tangent to find the angles correſponding thereto to a great exactneſs.

Fig. 3

381. Let E F G be ſuch a curve, that if the ordinates B E, C F, D G, to the axis A D, expreſs any ſines or tangents, the diſtances A B, A C, A D, expreſs the correſponding angles to theſe ſines or tangents; from the extremity E of the leaſt ordinate draw a line parallel to the axis A D, meeting C F in I, and D G in H; then if the interval B D between the extremes is not above a minute, the differences between the ſines will be proportional to the differences between the angles nearly.

That is E H : H G :: E I : I F, nearly; and if F L be drawn parallel to E H, there will alſo be E H : H G :: F L : L G, nearly.

Examp. I. Let the angle A C be 30° : 50′ : 50″; then if A B be 30° : 50′, and A D 30° : 51′; becauſe the ſine B E is 5125425, the ſine D G, 5127922, and their difference H G, 2497; B D or E H will be one minute or 60 ſeconds, and C D or F L, 10: Therefore E H (60) : H G (2497) :: F L (10) : L G (416); this being ſubſtracted from D G, gives 5127506, for the ſine C F required.

It muſt be obſerved, that if the ordinate C F is nearer D G than B E, the difference L G is to be found and ſubtracted from the greater D G, in order

der to get CF; but if CF is nearer to BE than DG, the difference IF is to be found and added to the leſſer BE.

Examp. II. Let 4567890 be the ſine whoſe correſponding angle is required; as this ſine is greater than 4565804, that of 27°: 10′; and leſs than 4568392, that of 27°: 11′; if we ſuppoſe BE to be 4565804, DG, 4568392, then will GH be 2588; LG, 502; and therefore HG (2588): EH (60) :: LG (502): FL (11.64); this being ſubtracted from EH (11′), gives 10′: 48.36″. Conſequently, AC, 27°: 10′: 48.30″, will be the angle required.

The ſame rule is to be obſerved for the tangents or their logarithms. Having thus ſhewn the uſe and conſtruction of the tables, we ſhall now proceed to the ſolution of triangles.

Solution *of* Right angled Triangles.

THEOREM II.

382. If, in any right angled triangle CDE, *Fig.* 4. from one of the acute angles C as center, the arc AM be deſcribed with any radius, and the tangent AT to that arc be drawn, as likewiſe the ſine PM; then by reaſon of the ſimilarity of triangles * we have CP: PM :: CA: AT :: CD: DE, and * *art.* 69. CP: CM :: CA: CT :: CD: CE. Which ſhews, *that the coſine is to the ſine, or the radius is to the tangent, as the adjacent ſide* CD *to the angle, is to the oppoſite ſide* DE.

And the coſine is to the radius, or the radius is to the ſecant, as the adjacent ſide CD *is to the hypothenuſe* CE.

CASE I.

383. *Let the ſides* CD, DE, *of a triangle right* *Fig.* 5.

angled at D *be given, to find the acute angles at* C *and* E.

If CD = 100, and DE = 60, then as CD (100) is to DE (60) so is the radius 100000 to the tangent 60000 of the angle C: Now looking into the tables of the fines and tangents for the angle corresponding to that tangent 60000, we fhall find 30°: 58′, nearly. Therefore the adjacent angle C is 30°: 58′, and the angle E, the complement to the angle C is 59°: 2′.

CASE II.

384. *Let one of the fides* CD *and an acute angle* C, *be given, to find the other fide* DE.

If the fide CD = 100, and the angle C 40 degrees; then the radius 100000 is to the tangent 83909 of 40 degrees, as the fide CD (100) is to the fide DE (83.909) required.

CASE III.

385. *Let one of the acute angles* C *and one fide* CD *be given, to find the hypothenufe* CE.

If the angle C be 40 degrees, and the fide CD = 100; then the radius 100000 is to the fecant of 40 degrees, or of the angle C, as the fide CD (100) is to the hypothenufe CE (130.54) required.

Thefe are all the cafes that can happen in right angled triangles; for when one of the acute angles is given, the other, which is its complement, will alfo be given; and when the two fides are given, the fum of their fquares is * equal to the fquare of the hypothenufe.

* *art.* 49.

THEOREM III.

Fig. 6, 7. 386. *In any triangle* ABC, *the fides are to each other as the fines of the oppofite angles.* If

If from any of the angles B, there be drawn a perpendicular BE to the oppoſite ſide AC; then if the radius be called R, and the ſines of the angles A, C; $\int A$, $\int C$,

We have* $\left\{\begin{array}{l}\int.A:BE::R:AB\\ \int.C:BE::R:BC\end{array}\right\}$ Therefore *$\int.A:\int.C::BC:AB$: *art. 382. * art. 85.

THEOREM IV.

387. *In any triangle where a perpendicular* CD *is drawn to the baſe, the baſe* AB *is to the ſum of the ſides* AC, BC, *as their difference is to the difference between the ſegments* AD, DB. Fig. 8.

From the vertex C as center deſcribe an arc of a circle with a radius equal to the leſſer ſide CA, interſecting the baſe in G, and the ſide BC produced in F and E. Then as CA, CE, CF, are radii, they are * equal; and becauſe the perpendicular CD * biſects the chord AG, BG will be the difference between the ſegments AD, DB; BF the difference between the ſides CB, CA, and BE their ſum. But by the property of the circle we have * AB : BE :: BF : BG; that is, the baſe AB is to the ſum of the ſides BC, AC, as their difference is to the difference between the ſegments. * art. 96. * art. 105. * art. 116.

THEOREM V.

388. *In any triangle* ABC, *the ſum of the ſides* AC, BC, *is to their difference as the tangent of half the ſum of the oppoſite angles* A, B, *is to the tangent of half their difference.* Fig. 9.

Let there be the ſame conſtruction as before, draw AE, AF, and FG parallel to EA; then as the external angle ECA is equal to the two internal * oppoſite ones A, B; and the angle EFA, which is at the circumference, is * half the angle * art 8. * art. 109.

ECA at the center; the angle EFA is half the sum of the angles A and B; and since the angles CAF, CFA, in the isosceles triangle ACF, are equal, the angle FAG, which is the difference between the greater angle CAB and half the sum CAF or CFA, will be half their difference.

Now because the angle FAE is contained in a
* *art.* 114. semi-circle, it will be a * right angle; and since FG is parallel to AE by construction, the angle AFG will also be a right angle: If therefore AF be taken for the radius, AE will be the tangent of AFE, half of the sum, and FG the tangent of half the difference FAG, of the angles A and B; and by reason of the parallels AE, GF, we
* *art.* 67. have * BE:BF, or BC+AC:BC—AC::AE:GF.

Solution *of* Oblique Triangles.

CASE I.

Fig. 10. 389. *Two sides of any triangle* ABC *being given, together with an angle opposite to one of these sides, to find the rest of the angles and side.*

If AC = 100, AB = 60, and the angle B 100 degrees, then as 80 degrees is the supplement of
* *art.* 386. 100, if we say * as AC (100) is to AB (60) so is the sine 98480 of 80°, to the sine 59088 of the angle C, which therefore is 36°:13′; and subtracting the sum 136°:13′, of the two known an-
* *art.* 12. gles B, C, from 180, we shall have * 43°:47′ for the angle A.

Whence the sine 98480 of the angle B, is to the sine 69193 of the angle A, as the side AC (100) is to the side BC = 70.26.

N. B. If the acute angle A is given, and the sides AB,

AB, BC, then if the side BC is less than BA, the angle C opposite to the side AB, may either be acute or obtuse; for there may be drawn another line equal to BC, such that the angle it makes with AC will be the supplement to the angle BCA; and therefore it must be known whether the angle sought is acute or obtuse before this case can be solved.

CASE II.

390. *Two angles and a side of any triangle being given, to find the other sides.*

If the angle A is 25 degrees, the angle B, 80, and the side AB, 120; subtract the sum 105 of the two given angles A, B, from 180 degrees, then the difference 75 will be the value of the angle C; therefore the sine 96592, of the angle C (75) is to the sine 98480 of the angle B (80) as * the side AB (120) is to the side AC = 122.34 *art. 386.

And the sine 96592 of the angle C (75) is to the sine 42261 of the angle A (25) as the side AB (120) is to the side BC = 52.5.

CASE III.

391. *The three sides of any triangle being given, to find the angles.*

If the side AC = 120, BC = 80, and AB = 100; then the * base AC (120) is to the sum 180 of the sides AB, BC, as their difference 20 is to the difference between the segments AD, DC, which is 30. This being added to the base 120; then half the sum which is 75, will be the value of the greater segment AD. *art. 387.

Now in the right angled triangle ADB, having the sides AD and AB, the angle A is found by saying, * as AD (75) is to AB (100) so is the radius *art. 382.

radius 100000, to the secant 133333 of the angle A, which will be 41 degrees and 25 minutes.

And in the triangle ABC, the sides AB, BC, being given, the angle C will be found by saying as BC (80) is to AB (100) so is the sine of the angle A (66153) to the sine 82691 of the angle C, which therefore is 55 degrees and 47 minutes; and the difference 82°: 48′, between the sum 97°: 12′ of the angles A, C, and 180 degrees will be the value of the angle B.

CASE IV.

392. *Two sides with the including angle of any triangle being given, to find the other angles and side.*

Let AC = 100, AB = 60, and the angle A, 53°: 48′; then as * the sum 160 of the sides AB, AC is to their difference 40, so is the tangent 197110 of 63°: 6′ half the sum of the opposite angles B, C, is to the tangent 49277 of half their difference; this answers to an angle of 23°: 54′.

* *art.* 388.

Now if we add this angle 23°: 54′, to half the sum 63°: 6′, we shall have 87°, for the greatest angle B; and subtracting 23°: 54′ from 63°: 6′, we shall get 39°: 12′, for the lesser angle C.

Hence the sine 99862 of the angle B (87) is to the sine 80696 of the angle A (53°: 48′) so is the side AB (100) to the side BC = 80.807.

These are all the different cases that can happen in any plane triangle; and to shew their use we shall add the following problems, which most commonly occur in practice.

PROB. IV.

393. *To find the distance from an inaccessible object*

Fig. 12.

object A, *placed beyond a river, or precipice, to a given point* B.

On a level ſpot of ground meaſure the baſe B C, place the inſtrument at B, and take the angle C B A; carry the inſtrument to the other extremity C of the baſe, and take alſo the angle A C B: Then having the baſe B C of the triangle A B C, and the adjacent angles B and C, the diſtances A B, A C, may be found by caſe I.

If the angle B be 70 degrees, the angle C, 68, and the baſe B C, 120 yards, by ſubtracting the ſum 138 of the angles B and C from 180, the difference 42 will be the value of the angle A. Therefore, the fine 66913 of the angle A (42) is to the fine 92718 of the angle C (68) as the baſe B C (120) is to the diſtance A B = 166.27 yards.

If the triangle A B C be laid down by a ſcale of equal parts and a protractor, and the ſides A B, A C, being carried on the ſcale with your compaſſes, they will be found pretty near the ſame as by computation.

PROB. V.

394. *To find the diſtance between any two inacceſſible objects* A *and* D.

In a convenient place meaſure the baſe B C, as likewiſe the angles made by that baſe and the lines drawn from its extremities to theſe objects; then in the triangle A B C, having the baſe and the adjacent angles, the ſide A C may be found by caſe I. And in the triangle B C D, having the baſe B C with the adjacent angles, the ſide C D may be known. Laſtly, in the triangle C A D, having the ſides C A, C D, with the included angle, the inacceſſible diſtance A D may be found by caſe IV.

If

If the base BC = 125 yards, the angles ACD (50), ACB (42), ABC (86), CBD (40); then the difference between the sum 128 of the angles ABC, ACB and 180 gives 52 for the angle BAC; and therefore, the sine 78801 of the angle A (52) is to the sine 99756 of the angle B (86) as the base BC (125) is to the side CA = 158.24.

The sum 132 of the angles BCD, DBC, being subtracted from 180 gives 48 for the angle CDB; whence, the sine 74314 of the angle D (48) is to the sine 64278 of the angle B (40) as the base BC (125) is to the side CD = 108.

Lastly, the sum 266.24 of the sides CA, CD, is to their difference 50.24, as the tangent 214450 of 65 half the sum of the angles A, D, is to the tangent 40310 of half their difference, which is 21°: 58′; this being added to and subtracted from half the sum 65, gives 86°: 58′, for the greater angle CDA, and 43°: 2′, for the lesser CAD. Consequently, the sine 68242 of the angle CAD (43°: 2′) is to the sine 76604 of the angle ACD (50) as the side CD (108) is to the distance required AD = 121.

PROB. VI.

395. *To find the height of an accessible object* AB. *Fig.* 13.

Place the instrument at a convenient distance from the object as at C; let CD be the height of the Instrument, and observe the angle BDE made by the horizontal line DE and the line DB; measure the distance CA or DE; then as all the angles in the right angled triangle DEB are known, as also the base DE; the perpendicular EB * may be found, to which adding the height AE of the point E from the Ground, and you will have the height required. Let

* *art.* 384.

Let the base C A or D E be 100 feet, and the angle E D B, 60 degrees; then as the radius 100000 is to the tangent 173205 of the angle D (60) so is the side D E (100) to the perpendicular E B (173.205) to which adding 4 feet for the height of the point E from the ground you will have A B = 177.205.

PROB. VII.

396. *To find the height of an inaccessible object* A B.

Place the instrument in a convenient place as at C, and observe the angle B D E, made by the horizontal line D E and the line D B; then go backwards in a direct line with the points D and E, at a convenient distance as at F, and take the angle B F E and measure the distance F D: Then having all the angles of the triangle F D B and the base F D, the side D B may * be found; and having the side D B and the angle B D E, the height E B may * be found.

* *art.* 390.

* *art.* 384.

If the angles B D E (62°: 50′), B F D (48°: 30′) and the base D F, 100 feet; then because the exterior angle B D E is equal to the sum of the two interior opposite ones F and B; the difference 14°: 20′, between the angles B D E, B F E, will be the angle F B D. Therefore, the sine 24756 of the angle B (14°: 20′) is to the sine 74895 of the angle F (48°: 30′) as the base F D (100) is to the side D B = 302.53; and the radius 100000 is to the sine 88968 of the angle B D E (62°: 50′) as the side D B (302.53) is to the perpendicular E B = 269.15; to which adding 4 feet for the height of the point E from the ground, you will have A B = 273.15 for the height wanted.

PROB.

PROB. VIII.

Fig. 11. 397. *The distance between three objects* A, F, B, *lying in a right line, being given together with the angles under which they are seen from any place* D, *to find the distances from this place to the three objects.*

Suppose a circle to pass through the two extreme objects A, B, and the point D; let a line be drawn from the point D and the third object F, meeting the circumference in E, and draw AE, BE; then because the angles EAB and
*art. 113. EDB, insisting on the same arc EB, are * equal, as well as the angles EBA and EDA, we have the three angles of the triangle AEB and the side AB, and so the side AE may be found; and having the sides AE, AF, with the included angle, the angle AFE may be found. Lastly, in the triangles ADF, BDF, having the angles at D and F, and the sides AF, FB, the required distances may be known.

If AB = 180 fathoms, AF = 50, the angle ADF, 21°: 15′, and the angle BDF, 38; then will the angle AEB be 120°: 45′; whence, the fine 85940 of the supplement 59°: 15′ to the angle AEB, will be to the fine 36243 of the angle ABE, or of its equal ADE (21°: 15′) as the side AB (180) is to the side AE = 75.9 or 76 fathoms. Now the sum of 126 of the sides AE, AF, is to their difference 26 as the tangent 290421 of 71 half the sum of the angles E, F, is to the tangent 59928 of half their difference, which answers to an angle of 30°: 56′, this added to 71, gives 101°: 56′ for the greatest angle AFE or BFD.

Therefore, the fine 61566 of the angle FDB (38) is

is to the fine 97838 of the ſupplement 78°: 4′, to the angle DFB (101°: 56′) ſo is the ſide FB (130) to the ſide DB = 206.5.

And the fine 61566 of the angle FDB (38) is to the fine 64367 of the angle DBF (40°: 41′) ſo is the ſide FB (130) to the ſide FD = 136.

Laſtly, the fine 36243 of the angle ADF (21°: 15′) is to the fine 97838, of the angle AFD (78°: 4′) ſo is the ſide AF (50) to the diſtance AD = 135.

This problem is very uſeful in the attack of a place; for by means of Dr. *Hadley*'s reflecting quadrant, the points of the baſtions before the attack may be ſeen as well as one of the ſhoulders, by looking between two ſand bags; the diſtances from one point of the baſtion to that of the next, as well as to the ſhoulder, are generally known from the author's method that fortified the place; and from thence, the diſtance from the trenches to the place are eaſily known; for though theſe three points are not exactly in a line, yet they are near enough to cauſe no great error.

If the point D is choſen in the capital of the ravelin produced, the line DF will be perpendicular to the exterior ſide AB; then the problem will be much eaſier; ſince the radius will be to the ſecant of the angle DAB or DBA, as half the exterior ſide AB is to the ſide AD or BD.

PROB. IX.

398. *The diſtances between three objects* A, B, C, *not placed in a right line being given, together with the angles under which they are ſeen from any place* D; *to find the diſtances from that place to the three objects.* Fig. 11.

Suppoſe a circle to paſs through any two of theſe objects A, B, and the point D, and let the line

line CD interſect or meet the circumference in E, draw AE and BE; then as the angles EAB, EDB, inſiſt on the ſame arc EB, they * are equal, as likewiſe the angles EBA and EDA; whence, in the triangle AEB, the ſide AB, and the adjacent angles are given, as being equal to the angles at D, therefore the ſide AE is alſo given: And becauſe the three ſides of the triangle ABC are given by ſuppoſition, the angles are * likewiſe given; and ſo the angle CAE, which is the ſum or difference of two given angles CAB and BAE, is alſo given; then as the ſides AC, AE, and the included angle are given, the angle ACE may * be found.

* *art.* 113.

* *art.* 391.

* *art.* 392.

Laſtly, having the ſide AC and all the angles of the triangle DAC given, the ſides DA and DC may * be found; and having in the triangle DBA, the ſides AB, AD, and the angles at A and D, the ſide BD may be * found.

* *art.* 390.

* *art.* 389.

If AB = 100, AC = 66, BC = 70, and the angles ADC, 27, CDB, 32, then the fine 85716 of the ſupplement 59 to the angle AEB is to the fine 45399 of the angle EBA(27) ſo is the ſide AB(100) to the ſide AE = 53: To find the angle CAB ſay, as the baſe AB(100) is to the ſum 136 of the ſides AC, BC, ſo is their difference 4 to the difference 5.44, between * the ſegments of the baſe, this being ſubtracted from the baſe 100; half the difference 47.28 will be the ſegment next to the angle A; but this ſegment 47.28 is * to AC(66) as the radius 100000 is to the ſecant 13959 of the angle A, which therefore is 44°: 14′; and hence the angle CAE is 12°: 14′, as being the difference between the angles BAC(44°: 14) and BAE, or its equal BDC(32). Now as the ſum 119 of the ſides AE,

* *art.* 384.

* *art.* 386.

AE, AC, is to their difference 13 so is the tangent 933154 of 83°: 53′, half the sum of the angles C, E, to the tangent 101941 of half their difference; which answers to an angle of 45°: 33′; this being subtracted from half the sum 83°: 53′, gives 38°: 20′ for the least angle A C D.

Therefore, the sine 45399 of the angle A D C (27) is to the sine 62023 of the angle A C D (38°: 20′), so is the side A C (66) to the side A D = 90.16. And as the sine 45399 of the angle A D C (27) is to the sine 90875 of the supplement 65°: 20′ to the angle D A C, so is the side A C (66) to the side D C = 132.2. Lastly, in the triangle D A B, the sine 85716 of the angle A D B (59) is to the sine 94225 of the angle D A B (70°: 26′), so is the side A B (100) to the side D B = 110.

If the point C falls any where in the circumference, the problem is indetermined; for the angles C A B, C B A, are in this case equal to the alternate angles at D; and so any point in the arc A D B will satisfy the problem.

This last problem is useful in surveying the plan or map of a country; for having once the distances between three towns, those to any other from which these are seen, may be found; and in this manner the distances between all the towns may be found one after another.

TRIGONOMETRY *applied to* FORTIFICATION.

When a plan of a fortification is to be traced on the ground, the masonry to be computed, or an estimate is to be made, it is necessary to know exactly all the lines and angles which compose it; for which reason we shall give the following spe-

cimen, applied to a front conſtructed according to Mr. *de Vauban*'s firſt method, which being well underſtood, may eaſily be applied to any other conſtruction whatſoever.

PROB. X.

Plate XIV *Fig.* 9. 399. *To find the lines and angles of the front of a fortification, conſtructed according to Mr.* de Vauban's *firſt method.*

The exterior ſide A B is ſuppoſed to be 180 fathoms or toiſes; the perpendicular C D, 30; the faces A E, B H, 50; and the lines E H, E G, and F H, are equal; we are from theſe data's to determine the reſt of the lines and angles.

The firſt thing to be found is the angle C A D, made by the exterior ſide and the faces of the baſtions: As in the right angled triangle A C D, we have the ſides AC(90) and CD(30) the angle A will be found to be 18 degrees and 26 minutes; twice this angle, ſubtracted from the angle of the polygon, gives the flanked angle of the baſtion: In an exagon, the angle of the polygon is 120 degrees; and therefore the flanked angle is 83 degrees and 8 minutes.

The ſide A D is to be found next, which is 94.87 fathoms; and ſubtracting the face A E(50) from A D, we ſhall have 44.87 for the line E D; whence, the ſimilar triangles A C D, E P D, give 42.57 fathoms for E P; and therefore E H and E G, which are double of E P, will each be 85.14 fathoms.

Now in the iſoſceles triangle E H F, having the angle E H F equal to the angle C B D (18°: 26′) the angles at the baſe E F will each be 80 degrees and 47 minutes; whence, having the two ſides E H, F H, and all the angles, the flank E F will be

Plate XII. p.

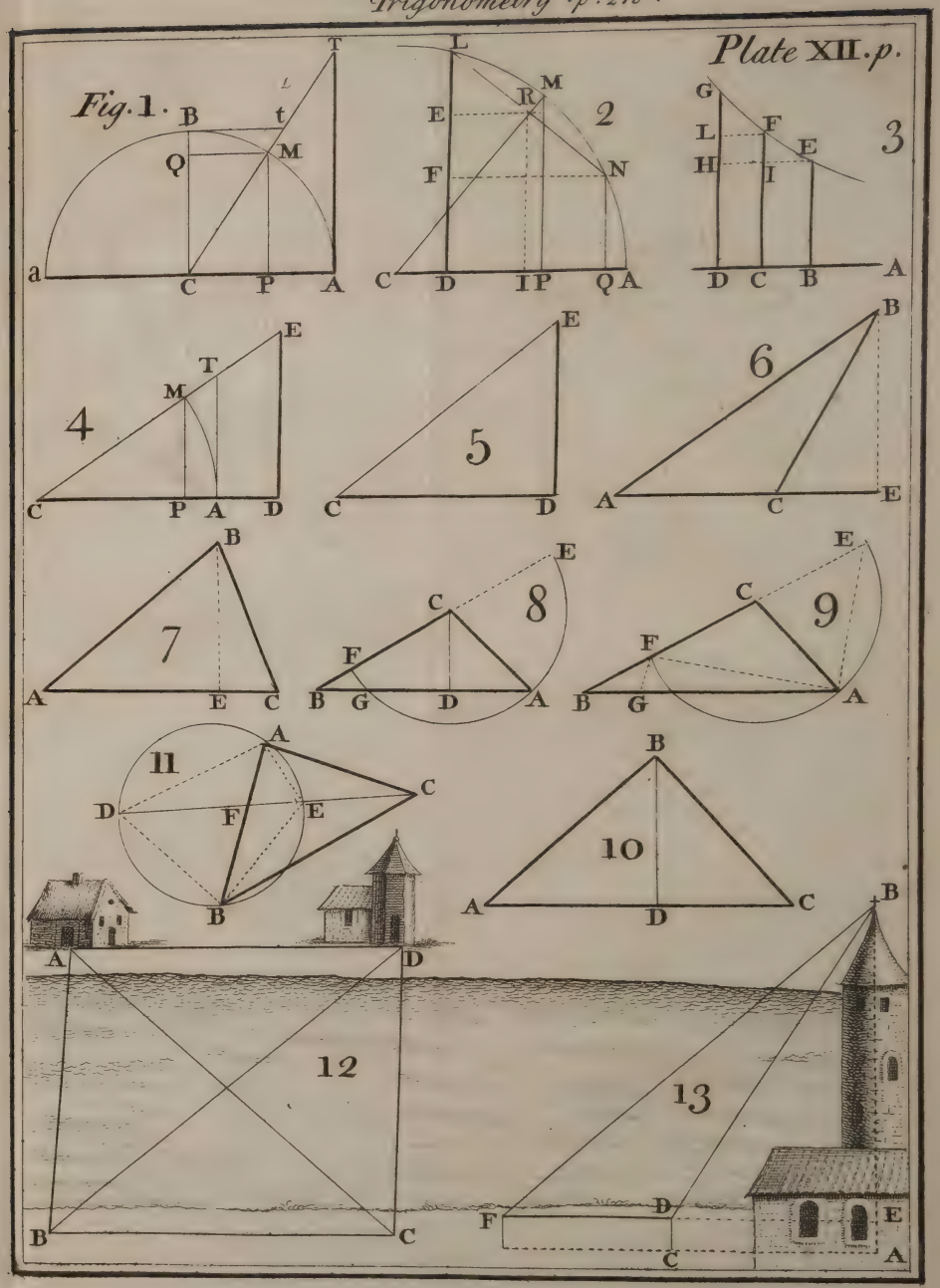
Fig. 1.
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be found to be 27.27 fathoms. But in the triangle G F E, having the sides E F, E G, the angle F G E, equal to its alternate one D A C, (18°: 26′) and the angle G F E at the curtain is equal to the sum of the angles H F E (80°: 47′) and G F H (18°: 26) and so is 99 degrees and 13 minutes, and the angle F E G 62 degrees 21 minutes; by which the length of the curtain F G is found to be 76.39 fathoms; and subtracting the angle F E G of 62 degrees and 21 minutes from two right angles, we shall have 117 degrees 39 minutes for the angle A E F of the shoulder.

These are all the lines and angles which compose the body of a place; but as they are often made with orillons and retired flanks, we shall shew how they are to be found.

PROB. XI.

400. *To find the length of the arc* R S, *which forms the retired flank.*

This flank is constructed in the following manner; the strait flank G H is found as usual, and divided into three equal parts, so that G T is two of them, and T H one; from the opposite flanked angle A, a line A T is drawn, in which the part T S is taken equal to 5 fathoms; and in the line of defence A G, produced, the part G R is taken equal to T S; then on the base R S is described an equilateral triangle R p S, and from the opposite angle p, as center, the retired flank is described.

As the face A E is 50 fathoms by construction, and E G has been * found to be 85.13, the side A G will be 135.13 fathoms; now in the triangle A G T, having the sides A G, G T, with the included angle A G T, of 80 degrees 47 minutes,

* *art.* 399.

*art. 392. the angle A T G will be found * to be 91 degrees 29 minutes, and the angle G A T 7 degrees 44 minutes; and from thence the side A T is found to be 133.36 fathoms.

If to A T we add T S, or 5 fathoms, we get 138.38 for A S; and if to A G (135.13) we add G R (5) we have A R equal to 140.13 fathoms. Now in the triangle R A S, having the sides A R, A S, with the included angle A of 7 degrees 44 minutes, the angle A R S at the base will be found

*art. 392. * to be 80 degrees 45 minutes; and having the side A S, and the angles of the triangle A R S, the side R S is found to be 18.86 fathoms.

As the radius R p is equal to the chord R S by

*art. 352. construction; if we say as * unity is to 3.1416, so is the radius p R (18.86) to its semi-circumference, we shall have 59.25 for the semi-circumference of the radius p R: And since the arc R S is 60 degrees, or one third of two right angles, the retired flank R S will be 19.75 fathoms.

PROB. XII.

401. *To find the orillon* T H, *according to Mr.* de Vauban's *construction.*

If from the extremity H of the face B H of the bastion, a perpendicular H n is drawn, and another which bisects T H at right angles; their intersection n is the center from which the orillon is described.

*art. 399. If from the angle of the shoulder * B H G (117°: 39′) we subtract the right angle B H n, we get 27°: 39′, for the angle n H r; and as T H is 9.09 one third of the flank G H, half of this gives 4.545 for H r: Therefore, in the right angled triangle H r n, having one side H r and all the angles, the radius H n of the orillon will be found

to

to be 5.13 fathoms. Now if we ſay as unity is to 3.1416 ſo is the radius H n(5.13) to its ſemi-circumference 16.116 fathoms; and as the angle H n r is 62°: 21′, twice that angle 124°: 42′ or 124.7 will expreſs the orillon in degrees.

If therefore we ſay as 180 degrees is to 16.116 fathoms, ſo is 124.7 degrees to the orillon 11.16 fathoms.

PROB. XIII.

402. *To find the length of the counterſcarp* I L R.

The arc L R or round part of the counterſcarp before the flanked angle of the baſtion, is deſcribed with a radius of 20 fathoms; and the counterſcarp produced meets the ſhoulder H. If A L be drawn perpendicular to I L, and joining the line A H, we have in the right angled triangle A L H the ſide A L and the right angle, which is not ſufficient to determine the reſt; for which reaſon the ſide A H muſt be found by ſome other triangle.

In the triangle A E H, having A E, 50 by conſtruction, and E H equal * to 85.14, with the included angle A E H, the ſupplement to G E H of 18 degrees 26 minutes, we ſhall find the angle A H E to be 6 degrees 48 minutes; and from thence the ſide A H comes out to be 133.53 fathoms. Now, in the right angled triangle A L H, having the ſides A H and A L, the angle A H L will be found to be 8 degrees 37 minutes; and its complement L A H, 81 degrees 23 minutes. Hence the ſide L H is 132 fathoms.

* *art.* 399.

In order to get the length I L of the counterſcarp, the part I H muſt be found, which is done by conſidering that in the right angled triangle H P I, the angle E H A has been found to be 6 degrees 48 minutes, and the angle A H L 8 de-

grees 37 minutes; so that the angle PHI is 15 degrees 25 minutes; and since the side PH is 42.56, the side IH will be found to be 44.15 fathoms; and this subtracted from LH (132) gives 87.85 fathoms for the strait part IL of the counterscarp.

To find the arc LR, or half the circular part before the bastion, it is necessary to know the external angle RAC, which in an exagon is 120 degrees; and as the angle HAC is equal to the alternate angle AHE, 6 degrees 48 minutes; by adding this angle to the angle CAR (120) we shall have 126 degrees 48 minutes for the angle HAR, from which subtracting the angle HAL 81 degrees 23 minutes, we shall have 45 degrees 25 minutes for the angle LAR; or because $\frac{25}{60}$ is nearly equal to .4171 decimals of a degree, the arc LR will be 45.417 degrees.

Now if we say as unity is to .31416 so is the radius AL (20) to its semi-circumference, 62.832; and then as 180 degrees is to 62.832, so is 45.417 degrees to the arc LR = 15.853 fathoms.

PROB. XIV.

403. *To find the several parts of the ravelin* NQM.

We suppose the capital IQ to be 50 fathoms, and the faces produced to meet those of the bastions, in a point O within three fathoms from the shoulder E. Hence, if we find DI we shall have in the triangle DQO the sides DQ and DO, with the included angle, by which the rest is determined.

*art. 402. *art 399. As the angle PHI was found to * be 15 degrees 25 minutes, and the side PH * 42.56 fathoms, the side PI will be 11.74 fathoms; and by

by the similarity of the triangles A C D, E P D, we get P D = 14.19 fathoms; this being added to P I (11.74) gives 25.93 fathoms for D I; and if to D I we add the capital I Q (50) we get 75.93 fathoms for the side D Q. The side D E has been found * to be 44.87 fathoms, to which adding E O (3) gives 47.87 for D O: And since the angle A D C is the complement to the angle D A C * 18 degrees 26 minutes, that angle will be 71 degrees 34 minutes, the angle D Q O will be found to be 36 degrees 45 minutes; and therefore the saliant angle N Q M of the ravelin is 73 degrees.

* art. 399.

* art. 399.

The angle P H I was found * to be 15 degrees 25 minutes, its complement P I H, or N I Q, will be 74 degrees 35 minutes; and therefore, in the triangle I N Q, having the side I Q (50) by construction and the adjacent angles, the side Q N or the face of the ravelin will be 51.75 fathoms; the semi-gorge I N, 32.12; and the angle I N Q, 68 degrees 40 minutes.

* art. 402.

It remains now to find the counterscarp of the ditch before the ravelin, in order to which draw Q T, Q V, perpendicular to these counterscarps; then as they are parallel to the faces of the ravelin, the lines Q T, Q V, are also perpendicular to the faces; and since the four angles at Q are * equal to four right angles, and the angles N Q V, M Q T, being right ones, the angle T Q V will be 107 degrees, the complement to the angle N Q M, 73 degrees. Whence, if we say as unity is to 3.1416, so is the radius Q T (12) to the semi-circumference 37.6992; and again, as 180 degrees is to 37.6992, so is 107 degrees to the arc T V = 22.46 fathoms.

* art. 4.

If N *b* be drawn perpendicular to the counter-

ſcarp *a*V, the part *b*V will be equal to the face NQ(51.75) fathoms, and the angle N *a b* will be 68 degrees 40 minutes equal to the angle INQ, its complement *a*N*b*, 21 degrees 20 minutes. Whence the ſide N*a*=12.88, and *ab*=4.36 fathoms.

SECT. II.

Of SURVEYING.

OF all the inſtruments invented for the uſe of ſurveying, there is only the Plain Table, and the new improved Theodolite, that are worth mentioning, the reſt being more for ſhow than for any real uſe; for which reaſon we ſhall deſcribe the manner of uſing theſe two only.

The plain table is a rectangular board of about 20 inches long, from 15 to 18 broad, and about half an inch thick; it is made of fir, mahogony, or any other dry wood which does not warpe; about this board is a frame of about an inch broad, let into it, in order to faſten the paper; this inſtrument is ſupported like all others, by a three-legged ſtaff, having a ball and ſocket.

There is alſo a long braſs ruler with ſights, ſo placed as to anſwer one of the edges which ſerves to find the directions of objects. There is likewiſe a needle and compaſs fixed to it, and the degrees of a circle are marked on the frame of ſome; but theſe things are of no real uſe, for which reaſon I never have them made.

For, if at 12 o'clock a thin wire or pin is ſtuck upright

upright on the paper, its ſhadow gives the meridian line of the ground much better than the needle.

PROB. I.

404. *To ſurvey a ſpot of ground with a plain table.*

Before you begin it will be proper, though not abſolutely neceſſary, to go round the ground with ſome body who is acquainted with its boundaries, in order to obſerve the moſt convenient place for the ſtations, which ſhould always be as near to them as poſſible, ſometimes within and ſometimes without, according as the nature of the ground will permit it; thoſe are to be pitched upon where one can ſee fartheſt, forwards and backwards, and where the chain may be carried in a direct line from one to another.

This being premiſed, place the table at the ſecond ſtation B, lay the plan of it as nearly level as you can, place alſo two ſtation ſtaves, the one at A and the other at C; chuſe a point on the paper to repreſent the place B; turn the braſs ruler about it till you ſee, through the ſight, the ſtation A, and draw a pencil line along the edge; then turn the ruler ſtill about the point B, till you ſee the ſtation C, and draw another line along the edge of the ruler; theſe two lines will make the ſame angle on the paper as the three places A, B, C, make with each other on the ground.

This done, two men with a chain meaſure the diſtances from B to C, and from B to A; the foremoſt takes a dozen of arrows, or ſticks, and fixes one at every chain's length in the ground, which he that follows, holds the end of the chain near, till it is ſtretch'd, and then takes it up; and they continue thus till the fore part of the chain reaches the

ſtation

ſtation; if it exceeds it by ſome links they muſt be taken notice of; and the number of arrows taken up by him who follows the chain, will give the number of chains meaſured between the ſtations.

Care muſt be taken that the chain is carried in a ſtrait line, and as nearly horizontal as poſſible; he who follows the chain muſt direct him that precedes; and, on the contrary, the firſt by looking backwards will be able to direct himſelf: It is neceſſary to have theſe diſtances very exact, becauſe the exactneſs of the ſurvey depends chiefly thereon.

In the ſame time that the diſtances from one ſtation to another are meaſured, the offsets at every chain's end are alſo meaſured, or ſometimes at a leſſer or greater diſtance, according as the boundary is more or leſs crooked, and marked in a pocket-book, together with the turnings of hedges, lanes, houſes, ſtiles and gates, and the names of the adjoining fields; for all theſe things muſt be repreſented in the plan.

Suppoſe that the diſtance A B has been laid down on paper, draw perpendiculars to A B at every chain's length, on which ſet off the ſeveral offsets noted in the book, and through their extremities the boundaries are drawn ſo as to repreſent a hedge, ditch, lane, or whatever elſe is on the ground.

This being done, the ſtation-ſtaff at A is carried to B, and ſome mark left there to find the place again, if required; the inſtrument is removed to the ſtation C, and another ſtation at D; the table being laid level, or nearly ſo, it muſt be turned ſo as the line C B on the paper be exactly in the direction of that line on the ground; then the braſs

ruler

ruler is turned round the point C, till the fourth ſtation D is ſeen through the ſights; draw the line C D; then the line C D being meaſured with the ſame care and exactneſs we mentioned before, as likewiſe all the offsets, and notice taken of every remarkable object; and when theſe things are marked on the paper, the inſtrument is to be removed to the fourth ſtation D, the ſtaff to C, and another to E.

The inſtrument being laid level, and turned ſo as the laſt line C D on the paper be in the direction of that line on the ground, the braſs ruler is turned about the laſt point D, till the next ſtation E is ſeen through the ſights; then draw the line D E; proceed in the ſame manner at every ſtation, till you come to the laſt, where you direct the line to the firſt ſtation A, and if the diſtance between the firſt and laſt ſtation anſwers exactly on the paper, you are certain that the ſurvey is rightly performed; but if it does not, ſome errors have been committed, either in taking the angles, or in meaſuring the diſtances.

If the ground be divided into ſmall fields, and there are any buildings, ponds, or rivers, in ſuch a caſe it is beſt to take the boundary of the whole at firſt, if it can be brought into one ſheet of paper; which being found exactly and laid down, the internal parts are ſurveyed afterwards, by ſetting out from ſome of the former ſtations: But if the extent of the ground is conſiderably large, ſo as not to be taken all on one ſheet, it will be more convenient to take the fields and other objects one after another, till the whole is finiſhed.

If there are but a few objects within the limits of the ſurvey, ſuch as trees, buildings, and gardens, in the manner marked at P; then from any ſtation

ſtation as G, where the corner of the houſe may be ſeen, you direct the ruler to it, and draw a line; and if you do the ſame thing from ſome other ſtations as H, I, K, the interſections of theſe lines will give the ſituation of the building on the plan.

If there be a river, ſuch as is marked here; from the next ſtation D you direct the ruler to a ſtation L; the diſtance from D to L being ſet off, you proceed thence along the river from L to M, and from M to N, quite to the end of the boundary; and as you go along take the offsets, eſpecially where the river bends moſt, and here and there the breadth, by which the courſe of the river will be determined.

Of the THEODOLITE.

THIS inſtrument is compoſed of a braſs circle 9 or 10 inches diameter, divied into degrees, called the Limb, and of a braſs ruler which turns about the center, having one end divided, according to *Nonius*, called the Index; and upon this index is fixed a vertical ſemi-circle divided into degrees, with a teleſcope ſliding up and down, having likewiſe an index under it divided according to *Nonius*. There is alſo fixed a box and needle to the index, with two ſpirit-levels at right angles to each other, to lay the plane of the inſtrument truly horizontal: The inſtrument is ſupported by a three-legged ſtaff in the ſame manner as the plain table; and inſtead of the ball and ſocket, four ſcrews, playing between two braſs plates, are uſed, which keep it much ſteadier and firmer.

In order to ſurvey with a theodolite, it is neceſſary to have a field-book divided into three columns,

columns; the middle one ſerves to mark the angles and diſtances from one ſtation to another, and the other two to inſert the offsets, either on the right or left, according as they are ſituated; the name and title of the land muſt be marked, with the pariſh where it lies, and the year it was ſurveyed.

The ſtations are maked thus ⊙ with the numbers 1, 2, 3, 4, *&c.* ſhewing their order affixed to them; and when it is neceſſary to return to any of the former ſtations, in order to cloſe ſome particular part, that ſtation is ſet down again with a number prefixed to it, ſhewing how often the inſtrument has been placed there; oppoſite to the number of degrees and minutes of the angles is marked a hook thus >, or thus <, to ſhew which way the angle turns, either to the left or to the right; for if this ſhould be neglected, it would not eaſily be known which way the next ſtation lies when the ſurvey is protracted.

PROB. II.

405. *To ſurvey with the new improved theodolite.*

Having made the neceſſary preparations, mentioned before, place the inſtrument at the firſt ſtation A, and level it by means of the croſs-levels and ſcrews underneath; fix the index to 360 degrees on the limb, turn the whole inſtrument round till the ſouth end of the needle hangs over the flower-de-luce, or 360 degrees, in the box, there fix the limb of the inſtrument by means of a ſcrew underneath, then diſcharge the index and turn it about till the vertical hair in the teleſcope cuts the ſecond ſtation B, there

fix

fix the index to the limb by means of a ſcrew, and the ſouth end of the needle will ſhew the bearing of the line drawn through the firſt and ſecond ſtations, which will likewiſe be cut by the index on the limb: This angle is to be entered into the field-book, under the firſt ſtation, with the hook > at the ſide of it, as has been mentioned before.

The diſtance A B between the firſt and ſecond ſtations being meaſured with a chain, in the ſame manner as has been mentioned in the laſt problem, as likewiſe all the offsets from and perpendicular to the line of direction to the ſeveral bendings of the boundary, and entered in the field-book, with notes in the margin of the hedges, ditches, ſtiles, gates, lanes, and houſes, which occur between theſe ſtations, and are to be expreſſed in the plan; the inſtrument is removed to the ſecond ſtation B, and after having levelled it, and ſo placed as the teleſcope cuts the firſt ſtation A; then the limb is fixed in that poſition; the index diſcharged, and turned about till the vertical hair of the teleſcope cuts the third ſtation C; there fix the index again and enter the number of degrees and minutes it cuts on the limb; the ſame angle will likewiſe be nearly pointed at by one end or other of the needle in the box, if no miſtakes have been made.

Care muſt be taken that the index has not been moved by carrying the inſtrument from one ſtation to another; this may be known by ſeeing whether the angle it cuts on the limb is the ſame with that marked in the field-book: It muſt likewiſe be obſerved, that when the index is moved, the limb of the inſtrument remains in its place; and when the limb is moved that the index moves with it, ſo as to point always at the ſame number of degrees.

The

The distance between the second and third stations being measured, as well as the offsets, and entered into the field-book, together with all the remarks concerning the objects which are to be inserted in the plan; the instrument is removed to the third station C, where the same operations are performed as at the second; that is, the instrument being laid level, and turned about till the vertical hair of the telescope cuts the second station B; the limb is fixed there and the index discharged, which being turned about till the vertical hair cuts the fourth station D; there it is fixed, and the angle it cuts on the limb is entered into the field-book, as well as the distance and offsets between the third and fourth stations, together with all the necessary remarks.

This being continued all round to the last station, where the instrument is turned to the first in order to close the work, and to find whether there has any mistakes been made in the taking of the angles; the instrument is placed again at the first station, and the angle it makes with the last and second is taken; if, which is the same or nearly so as that taken there before, you are certain that no errors have been committed in the taking of the angles.

When there is any occasion to return to a former station in order to proceed from thence for closing some particular fields, it is to be entered again in the field-book, as has been mentioned before; for which reason marks are left in those places where there is occasion to return again.

There is a great nicety required in keeping rightly the field-book, so as to mark distinctly all the particulars which are to be expressed in the plan, and in as short a manner as possible; and for

for the better underſtanding, there ought to be ſketches added to the remarks, which may be placed on ſome other leaf and referred with notes to the ſtations and offsets where they have been made.

Thus for example, the houſe and gardens marked P in the plan, being ſketched on a piece of paper, and the dimenſions of the building marked on it; then there is no more required than to take angles from three or four ſtations to the corners of the houſe; as ſuppoſe from the ſtations G, H, I, K, by which the place in the plan of this houſe will be determined.

In order to ſurvey the river, the inſtrument is placed at the fourth ſtation D, and the index turned till it cuts on the limb the ſame angle as at the third ſtation C; then after the inſtrument is levelled, and turned about till the vertical hair in the teleſcope cuts the third ſtation C, there the limb is fixed, the index diſcharged, and turned about till that hair cuts the ſtation L; the angle cut by the index on the limb being entered in the field-book under the ſtation D, with the number 2 prefixed to it, to ſhew that the inſtrument has been placed twice there; the diſtance from D to L being meaſured, and entered into the field-book, together with the proper offsets and notes, the operations are continued as before.

To render the preceding diſcourſe as intelligible as poſſible, we ſhall ſubjoin a ſpecimen of a field-book, in order to ſhew how the ſeveral particulars are to be entered, and afterwards how the plan is to be laid down from this book, by which it is hoped, that the intelligent reader may ſurvey by himſelf whenever occaſion requires it.

FIELD-

FIELD-BOOK of part of the manor, &c. *in the parish* M, *beginning at the lane* N, Feb. 16, 1747.

Remarks left	*Offsets*	⊙ *s.*	*Offsets*	*Remarks right*
		⊙ 1. A		
Bearing	>	75 : 15	SW	hedge
		650	100	hed. turns off
		1300	75	gate
		1650		
		⊙ 2. B		
	>	219 : 30		
		770	150	hedge
		1500	250	hedge
		⊙ 3. C		
	<	245 : 50		
		540	160	hedge
		1150	40	hedge
		⊙ 4. D		
	>	318 : 30		
		150	50	gate
		1550	350	to the river
		2652	30	gate
		⊙ 5. E		
	>	310 : 50		
		1240	25	hedge
		⊙ 6. F		
	<	322 : 40		
		900	30	hedge
		1920	25	hedge

Remarks left	*Offsets*	⊙ *s.*	*Offsets*	*Remarks right*
		⊙7. G		
bearing	>	305:0		
		750	230	hedge
		1100	140	hedge
		1470	260	hedge turns off
		⊙8. H		
	>	277:30		
		400	140	hedge
		1430	30	hedge
		⊙9. I		
	>	223:10		
		1100	150	hedge
		1860	45	gate, road to the farm
		⊙10. K		
		2 ⊙4. D		
	>	317:0		
		1420		
		⊙11. L		
		200	30	river
		800	150	
		1400	460	

N. B. The distances are marked in chains and links; the chain is 66 feet, or 4 poles, divided into a hundred links, invented by one *Gunter*, for which it is called *Gunter*'s Chain: The offsets are

are measured by a rod of ten links; this way of measuring the distances and offsets, is much better than with any other chain of yards or feet; because the several parts are set down with more ease, by means of a scale divided into 100 equal parts.

If there occurs several objects opposite to the same distance, which are to be taken, it will be better to set down that distance several times, by which all confusion is avoided.

PROB. III.

406. *To protract the observations on paper, or, which is the same, to draw the plan of a survey from the field-book.*

Draw a line at pleasure, representing a meridian, mark one end of it with a flower-de-luce, representing the south; place the diameter of the protractor on this line, so as the number 360 on the limb be opposite to the flower-de-luce, and there fasten it with some pins, or short nails: This being done, turn the brass ruler about till the index marks on the limb the angle 75 degrees and 15 minutes, taken at the first station; draw a pencil line along the edge of the ruler and mark it with the number one; then turn the ruler about till the index points at the angle 219 degrees 30 minutes, taken at the second station; draw a pencil line, which mark with the number 2, turn the index till it points at the angle 245: 50, taken at the third station, draw a line and mark it with the number 3.

Continue thus, in laying down all the angles, without moving the protractor; but if there are so many that some of the lines come too near one another, one or more meridians may be drawn

parallel to the firſt, and the protractor placed on them, by this means all the angles may be marked without confuſion.

Having marked all the angles, chuſe a convenient place for the firſt ſtation A; then laying the edge of a long parallel ruler on the line marked 1, and open it ſo till the ſame edge paſſes thro' the point A, or the firſt ſtation, where you draw a line, on which ſet off the diſtance 1650 between the firſt and ſecond ſtations; that is, 16 chains and 50 links, which gives the ſecond ſtation B; lay the edge of the ruler on the line marked with the number 2, and ſlide it ſo as the ſame edge paſſes through the point B, where you draw another line, on which ſet off the diſtance 1500, that is 15 chains, between the ſecond and third ſtations; this gives the third ſtation C; lay the edge of the ruler on the line marked 3, and ſlide it ſo till it paſſes through the point C, there draw a line and ſet off the diſtance 1150 between the third and fourth ſtations, which gives the fourth ſtation D: Continue thus to lay down all the ſtations to the laſt; then if the laſt line paſſes thro' the firſt ſtation, and the diſtance on the paper anſwers that in the field-book, you are certain that no miſtake has been made; but if it does not, and all the angles have been found right, on the ground, as we have mentioned in the article of ſurveying, ſome error has been committed in meaſuring the diſtances.

In order to diſcover where the miſtake lies, protract backwards, till you come to a line which falls on the ſame line which was found before by the firſt protracting; then you are certain that you are right; and this will always happen, if the angles are right.

The

The ſtations being laid down exactly, you begin at the firſt to mark the ſeveral offsets through which the boundaries are drawn, ſo as to repreſent hedges or ditches, according to the notes in the field-book.

Every thing being marked on the plan, it will be proper to go round the ſurvey with it, to ſee whether nothing has been omitted; and if there is any thing wanted, it muſt be inſerted by gueſs only, if not very material, or otherwiſe it muſt be meaſured.

PROB. IV.

407. *To find the number of acres contained in a ſurvey.*

As the boundaries are generally irregular, it is neceſſary to draw pencil lines, ſo as to reduce the plan into a rectilinear figure in ſuch a manner that the parts cut off are nearly equal to thoſe taken in; the reader may eaſily perceive that this cannot be done otherwiſe than by gueſs, but as a geometrical exactneſs is not required in this work, it may be done ſo near as to cauſe no conſiderable error.

This being done, divide the figure, by pencil lines, into ſquares, rectangles, and triangles, in the manner moſt convenient, and find the areas of the ſeveral parts ſeparately, add them into one ſum, and you will have the contents of the whole.

The area of a ſquare or rectangle will be found by multiplying their ſides together, expreſſed in numbers, and that of a triangle is half the * rectangle of the ſame baſe and altitude, or equal to half the product of its baſe and altitude. * *art.* 35.

A ſtatute acre is 40 poles long and 4 broad; that is, 4×40, or 160 ſquare poles; and becauſe the *Gunter*'s chain is 4 poles long, an acre is 10

chains in length and 1 in breadth; but this chain is divided into 100 links; 10 chains make 1000 links, which being multipled by one chain, or 100 links, gives 100000 for the content of an acre expreſſed by ſquare links. Which ſhews, *that if the area of a ſurvey is expreſſed in ſquare links, by ſtriking off the five laſt figures to the right as decimals, the remaining ones will expreſs the number of acres contained in the ſurvey.*

For example, if 176546 expreſſes the number of ſquare links contained in a ſurvey, by ſtriking off the five laſt figures 65436, we ſhall have 17 acres, and 65436 decimal parts of an acre, which being multiplied by 4, gives 2.51744 or 2 poles, four of which make an acre, or 2.5 nearly: Therefore the content of this ſurvey is 17 acres and 2.5 poles.

We have not given any figure to ſhew in what manner an irregular ſpot of ground may be reduced into a rectilinear figure, becauſe it is quite arbitrary, and the reader who has read the former part of this work will be able, upon occaſion, to perform it without any trouble.

PROB. V.

408. *To ſurvey or take the plan of a fortification.*

The manner of ſurveying a fortification does not differ from that deſcribed before; it is rather more eaſy, by reaſon of its regularity: Of all the different ways of proceeding, the following one appears to me the moſt ſimple and practicable. Place the inſtrument ſo as that you may take the angle made by the curtain and the flanks, take alſo the angle from that ſtation to the next, in the ſame baſtion near the other curtain; meaſure the curtain and flanks; and if you go round the rampart

part and continue the ſame operations near each curtain, you will get the plan of the body of the place: For becauſe the faces of the baſtions when produced generally meet the curtains at their extremities; therefore a line drawn through the extremity of the curtain and that of the oppoſite flank will give the direction of the face, and the line drawn through the extremity of the other flank of the ſame baſtion, and that of the adjacent curtain, will determine both faces.

If the faces produced do not meet the curtains at their extremities, the diſtance from the extremity of the curtain to the point where the face produced meets the ſame, muſt be meaſured; which being done, the reſt of the operations will be the ſame as before.

Having determined the body of the place, the next thing to be done is to find the counterſcarp of the great ditch; in order to which, obſerve whether it meets the ſhoulder of the baſtion when produced, as it generally does, then take the angle it makes with the flank, or the face of the baſtion; and if the place is regular, this will be ſufficient to determine the breadth of the ditch all round: For, if lines are drawn from the ſhoulders, ſo as to make the given angles with the flanks or faces, and from the ſaliant angles of the baſtions perpendiculars are drawn to theſe lines, they will be the radii of the round part of the counterſcarp.

If the place is irregular theſe angles muſt be obſerved all round; and if the counterſcarp does not meet the baſtions at the ſhoulders, the diſtance from that point to the ſhoulder muſt be meaſured, and then the reſt of the operations are the ſame as before.

The ravelines, the counterſcarp of their ditches, and the covert way, are all determined in the ſame manner as the counterſcarp of the great ditch; that is, the point where they meet the faces of the baſtions, or any other known part, being obſerved, and the angles they make with theſe parts meaſured then there is no more required than to draw lines through the given points, ſo as to make the given angle, in order to have the outlines of theſe works. As to the parapets and ramparts, they are always of the ſame breadth as thoſe of the body of the place.

If there are any outworks which do not meet the body of the place, when produced, they muſt be taken from without; the directions of the branches of horn and crown-works muſt be taken from within the place, and the other parts from without.

The gates of the town, magazines, ſtore-houſes, and barracks ought to be inſerted in the plan of a fortification; and if they are of any note, their front elevations ſhould be given on a particular ſcale: In ſhort, every thing belonging to the government ſhould be taken notice of.

PROB. VI.

409. *To take the map of a country.*

As the ſurvey of a large extent requires much more exactneſs and care than a ſmall one, it is neceſſary to have ſuch inſtruments as will meaſure the angles to a great nicety; for which reaſon a ſector or quadrant muſt be had, whoſe radius being about 5 feet, and ſo contrived as to meaſure ſeconds: For a few ſeconds, more or leſs, in large ſurveys, will cauſe conſiderable errors at the end of the work.

Having

Having proper inſtruments for ſuch an undertaking, a large plain is pitched upon, nearly in the middle of the country, ſo as a baſe of at leaſt a mile long may be meaſured; this ſhould be done with the greateſt exactneſs poſſible, and with which a triangle is to be formed to ſome ſteeple in a town; then the diſtance from one end of this baſe to the town is to be found by trigonometry; and with it, and any other adjacent towns, another triangle is to be formed ſo as to get the diſtance between theſe two places: And it is this laſt diſtance which is to be taken for the baſe, to form triangles from all the different towns. As the exactneſs of the whole ſurvey depends on this baſe, there cannot be too much caution uſed to meaſure it; for which reaſon I would chuſe to form ſome other triangle by means of a different baſe; then if the ſame meaſure is found for this diſtance, one may depend on its exactneſs.

Having the exact diſtance between any two places, it will ſerve as a common baſe to form from thence triangles to all the circumjacent places; and the diſtances between theſe places, as baſes to form other triangles to ſome others; and in this manner a chain of triangles may be formed, ſo as to go through all the conſiderable places of the whole country.

It may ſometimes happen, that from two ſtations, another cannot be ſeen; in ſuch a caſe a pyramid of wood or high poles are erected on ſome hill, from whence the triangles may be carried on; when the diſtances are very great, fires are lighted upon eminences, to guide the obſervations by, eſpecially if the objects cannot otherwiſe be ſeen.

As there cannot be too many cautions uſed, in order to avoid errors, it is neceſſary to find the latitudes

titudes of all the the principal towns, by obſervations. This may be done by obſerving the altitude of the ſun at 12 o'clock exactly; and by a table of the ſun's declinations, you find the elevation of the equator in that place; and the complement of this angle will be the latitude required. There are various methods for finding the latitude of a place, which thoſe who are employed in ſuch a work as this can, or ought, not to be ignorant of.

Having determined the exact poſitions of all the principal places, the next thing to be done is to determine the high-roads, and the courſes of great rivers: This may be done with a theodolite, in the ſame manner as has been ſhewn in prob. II. only there is no occaſion to take the offsets ſo frequently; it will be ſufficient to take the principal bendings. The plans of all the remarkable places ſhould be taken, and it ſhould particularly be obſerved, on which ſide of the principal ſtreets the church ſtands, becauſe the diſtances are determined by their poſitions.

If there are any conſiderable woods, their limits ſhould be determined; as likewiſe all the great roads which paſs through them; high hills muſt likewiſe be marked in the map; and if they are of an extraordinary height or ſize, ſketches are to be made of them, to be placed by themſelves in ſome corner of the map.

As in a country the variety of objects is infinite, ſo it is is impoſſible to give particular directions concerning the manner of obſerving them: It requires great skill in making the obſervations, and to chuſe proper places for the ſtations: But as works of this nature are never undertaken by unexperienced people, we hope what has been ſaid

on

on this ſubject will be ſufficient to ſuggeſt ſuch means to them as will be a great help in carrying on the work with ſucceſs.

SECT. III.

Of LEVELLING.

THERE are two ſorts, the water and ſpirit, or air-level; the firſt is commonly made of a glaſs tube of about two feet long, ſometimes ſhorter, of an inch diameter, the ends of which being bent at right angles for the length of two inches or thereabouts, and ſtopt with braſs ſtoppers, to prevent the water from running out.

This level is ſupported by a three-legged ſtaff, in the ſame manner as the theodolite, fixed to the middle of the tube with a braſs ring.

In the country, where a glaſs tube is not eaſily to be had, they make a tube of tin about the ſame dimenſions, to the ends of which are fixed two viols with ſealing wax, the bottoms being broke ſo as the water in the tube may have a free communication with that in the viols.

PROB. VII.

410. *To level with the water level.*

Place your level at a convenient diſtance from the place of your ſetting out, ſo as the water riſes in both ends of the viols; direct your eye over the two ſurfaces of the water, towards the firſt object, where ſome body muſt ſtand with a white ſheet of paper, to raiſe or depreſs it, according to the motion of your hand, along the ſtation-ſtaff, till the black

black line drawn acroſs the paper is cut by the line of obſervation; this being done, the height of this line from the ground is meaſured, and you go to the other end of the level and look forward, to the next ſtation, where the height of the black line from the ground is alſo meaſured.

The inſtrument being carried forward, as much beyond the ſecond ſtation as you can conveniently ſee, and another ſtation placed farther, you make the ſame obſervations backwards to the laſt ſtation, and forward to the next; and you continue in the ſame manner till you come to the object, whoſe level, in reſpect the firſt, is wanted.

This done, all the heights taken backwards being marked in one column, and all thoſe taken forewards in another, then the ſum of the one ſubtracted from the other, will give the difference between the levels of the firſt and laſt object: If the ſum of the heights forewards exceeds the ſum of thoſe backwards, the firſt object is higher than the ſecond, by the difference between theſe ſums, and on the contrary.

Becauſe of the roundneſs of the earth, the level found between any two objects in the preceding manner is not the true one, for which reaſon it is called the apparent level.

PROB. VIII.

Plate XIII. Fig. 3.

411. *To find the difference between the true and apparent levels of any two places*

Let O be the center of the earth, OA, OD, any two radii, AB a tangent to the ſurface ADE, at A: Now if the level be placed at A, it will give the apparent level AB, between the places A and D; whereas the true level is the arc AD: So that the apparent level exceeds the true one by the line BD:

For

For which reaſon this line muſt be found and ſubtracted from the apparent level, in order to get the true level between the places A and D.

By the property of the circle, we have * $\overline{2OD+DB} \times DB = \overline{AB}^2$; or, becauſe the diameter of the earth is ſo great in reſpect to the line BD, at any diſtance not exceeding 30000 yards, that 2OD may ſafely be taken for 2OD + DB, without any ſenſible error, as may be found by calculation: Therefore we have $2OD \times DB = \overline{AB}^2$, according to this ſuppoſition; or, $DB = \frac{\overline{AB}^2}{2OD}$. Which ſhews, *that the difference between the true and apparent level is equal to the ſquare of the diſtance between the objects, divided by the diameter of the earth.* * *art.* 116.

The diameter of the earth is 653869 toiſes or fathoms, *French* meaſure, according to Mr. *Picard*, or 13077388 yards; and as the *Engliſh* foot is to the *French* royal foot as 107 to 114, according to the moſt accurate compariſon, the diameter of the earth will be 13932824 yards *Engliſh* meaſure.

Now ſuppoſe we want to know the difference between the true and apparent level, at a diſtance of an *Engliſh* mile, or 1760 yards; the ſquare of this number is 3097600, which being divided by the diameter of the earth 13932824, expreſſed in the ſame meaſure, gives 0.222 decimals of a yard; or multiplying this number by 36, the number of inches in a yard, gives 7.999, or 8 inches for the ſaid difference. This agrees with what other authors ſay on this ſubject.

In order to ſave the trouble of computing every time the difference between the true and apparent level, we ſhall inſert the following table of corrections

rections from a diſtance of 210 yards to 1000, expreſſed in inches and decimal parts of an inch.

A TABLE, *ſhewing the corrections to be made between the true and apparent circle.*

210	220	230	240	250	260	270	280	290	300	Diſt.
.11	.12	.14	.15	.16	.17	.19	.20	.22	.23	Cor.
310	320	330	340	350	360	370	380	390	400	Diſt.
.24	.26	.28	.30	.32	.33	.35	.37	.39	.41	Cor.
410	420	430	440	450	460	470	480	490	500	Diſt.
.43	.45	.47	.49	.52	.54	.57	.59	.61	.64	Cor.
510	520	530	540	550	560	570	580	590	600	Diſt.
.67	.70	.72	.75	.78	81	.84	.87	.90	.92	Cor.
610	620	630	640	650	660	670	680	690	700	Diſt.
.96	.99	1.02	1.06	1.09	1.12	1.15	1.19	1.23	1.27	Cor.
710	720	730	740	750	760	770	780	790	800	Diſt.
1.30	1.34	1.38	1.41	1.45	1.49	1.53	1.57	1.61	1.65	Cor.
810	820	830	840	850	860	870	880	890	900	Diſt.
1.69	1.74	1.78	1.82	1.86	1.91	1.95	2.00	2.04	209	Cor.
910	920	930	940	950	960	970	980	990	1000	Diſt.
2.14	2.19	2.23	2.28	2.33	2.38	2.43	2.48	2.53	2.58	Cor.

By means of this table the apparent level may be correct, for any diſtance not exceeding 1000 yards: If the diſtance is not to be found exactly in this table, that which comes neareſt to it muſt be taken, for the difference will hardly be ſenſible.

Becauſe moſt ſurveyors uſe the *Gunter*'s chain in ſurveying and levelling, and compute the corrections by the diſtances expreſſed in chains, we ſhall

inſert

inſert the following table, where the diſtances are expreſſed in chains, and the corrections in inches and decimal parts of an inch.

5	6	7	8	9	10	11	12	13	14	Diſt.
.03	.04	.06	.08	.10	.12	.15	.18	.21	.24	Cor.
15	16	17	18	19	20	21	22	23	24	Diſt.
.28	.32	.36	.40	.45	.50	.55	.60	.67	.72	Cor.
25	26	27	28	29	30	31	32	33	34	Diſt.
.78	.84	.91	.98	.105	1.12	1.19	1.27	1.35	1.44	Cor.
35	36	37	38	39	40	41	42	43	44	Diſt.
1.53	1.62	1 71	1.80	1.90	2.00	2.11	2.21	2.30	2.42	Cor.

Theſe tables are computed in an eaſy manner by means of logarithms; for the logarithm of the diameter of the earth 13932824, expreſſed in yards, is 7.14404; no more figures are neceſſary; which being conſtantly ſubtracted from double the logarithms of the diſtances, the differences will be the logarithms of the corrections expreſſed in decimal parts of a yard; which being multiplied by 36, the number of inches in a yard, gives the corrections in inches and decimals of an inch.

Having explained the nature and uſe of the water level, as likewiſe in what manner the apparent level is corrected, we ſhall now give the deſcription of the ſpirit level, with the manner of correcting and uſing it.

Of the SPIRIT LEVEL.

This level is a glaſs tube of about 6 inches long, half an inch diameter, filled with ſpirits almoſt quite

quite full, ſo as to leave a little air in it; this tube is fixed to a teleſcope of about two feet long, with croſs hairs in it; and the whole inſtrument is ſupported by a three-legged ſtaff, in the ſame manner as the plain table or theodolite.

The ſtation ſtaves uſed in levelling are two rulers, each about 5 feet long, ſliding along each other by means of two brass ferules fixed to their ends; theſe rulers are divided into inches and tenths of inches; at one end is fixed a board of 6 inches long and 4 broad, called Vane, which ſlides likewiſe along the ruler, and ſerves to paſte a piece of paper upon it, one half white and the other black, placed horizontally.

PROB. IX.

412. *To rectify the ſpirit level.*

The firſt thing to be done is to fix the interſection of the croſs hairs exactly in the axis of the teleſcope; in order to which obſerve a point as far off as may be ſeen diſtinctly; then turn the teleſcope round its axis, till the lower part comes uppermoſt; and looking again to ſee whether the interſection of the croſs hairs cut ſtill the ſame point; if it does, the hairs are rightly fixed; but if not, ſlide the horizontal hair ſo as their interſection cuts in a point between the two former, and turn the teleſcope round its axis till the upper part is again the lower.

Obſerve again the point where the croſs hair cut an object; if it is the ſame as the laſt obſerved, the teleſcope is rectified; if not, ſlide the horizontal hair ſo as their interſection cuts a point between the two laſt obſerved, and continue in this manner till the interſection of the croſs hairs cut

cut always in the same point which way it is turned.

The next thing to be done is to bring the level parallel to the axis of the telescope; for which, set the instrument so as that the bubble in the glass tube be in the middle; observe through the telescope where the cross hairs cut an object; then take it off, and turn the instrument round its axis, so that the part which was next to the eye be towards the object; place the telescope on the instrument, and see whether the intersection of the cross hairs cut the same point as before; if not, raise one end of the telescope, or the level, by means of a screw made for that purpose; then set the instrument again, so as the bubble be in the middle of the glass tube, and observe where the cross hairs cut an object; take off the telescope; turn the instrument round into its former place, and see if the cross hairs cut still the same point; if not, raise or depress the telescope or level, and continue the same operations till the instrument is rectified.

When the level is fixed to the cross bar which supports the telescope, it is much better than when it is fixed to the telescope itself; for the brass tube being very thin, the screws cannot have hold enough to keep it firm: The forks which support the telescope, seem to me ill contrived, because the screws that fix them are not sufficient to support such a weight as the telescope and the forks together; besides, I think that the telescope does not lie so steady as it ought to do.

PROB. X.

Plate XIII. Fig. 2.

413. *To find the difference between the levels of any two places,* P *and* S, *which are at a great distance from each other.*

Having found a convenient place between the two firſt ſtations P and Q to place the inſtrument, raiſe it ſo as the air-bubble be exactly in the middle of the glaſs tube; direct the teleſcope to the firſt ſtation P, where a Perſon ſtands to raiſe or depreſs the vane, according to the motion of your hand, till the horizontal line which ſeperates the black part from the white is cut exactly by the interſection of the croſs hairs, which ſuppoſe to be at number 1; then the height of that point from the ground is marked in a pocket book, and you direct the inſtrument to the ſecond ſtation Q; ſeeing that the bubble remains in its proper place, if not, the level muſt be rectified; then obſerve the place 2, where the croſs hairs cut that ſtation; and having taken the height of that point from the ground, and marked it in the book, the diſtances from the level to each ſtation are likewiſe meaſured; this done, the inſtrument is removed to ſome convenient place B, between the ſecond Q and third ſtation R; ſet the level ſo as the air-bubble be in the middle of the glaſs tube, and direct the teleſcope back to the ſecond ſtation Q, the point 3, where the interſection of the croſs hairs cut the ſtation, being obſerved, and its diſtance from the ground entered in the book, turn the inſtrument and direct it to the third ſtation R, ſeeing that the bubble remains in the middle of the tube, the point 4, where the croſs hairs cut that ſtation being obſerved, and its diſtance from the ground entered; the diſtances from the inſtrument

ment to the ſtations being alſo meaſured and entred, the inſtrument is removed to a convenient place C, between the third and fourth ſtations, where the ſame operations are performed as before, and are continued to the laſt ſtation. This done, ſet all the heights obſerved backwards in one table, and all thoſe obſerved forewards in another; do the ſame thing in regard to the diſtances, from the inſtrument to the ſtations; correct the apparent levels by means of the foregoing tables, then ſubtract the leaſt ſum of all the heights corrected of one ſide, from the greater ſum of all the heights corrected of the other, and the remainder will be the difference between the heights required of the two places. If the ſum of all the heights foreward exceeds the ſum of all thoſe backward, the firſt place is the higheſt, as has been obſerved before.

EXAMPLE.

Diſt.	*Heights.*	*Cor.*
A 2 250	P 1 10	9.83
B 3 200	Q 3 3	2.90
C 5 360	R 5 8	7.69

Diſt.	*Heights.*	*Cor.*
A 2 350	Q 2 6	5.69
B 4 460	R 4 16	15.51
C 6 300	S 6 13	12.77

First sum	20.43
Second sum	33.97
Difference	13.54

Whence the first place P is 1920 yards distant from the second S, and is higher by 13,54 inches.

It may be observed, that if the distances from the level to both stations are equal, there needs no corrections to be made; but it is not always convenient to place the instrument in this manner.

It is not necessary that this level should be in a right line with the stations, as in the water level; for whether it makes any angle therewith or not, it signifies nothing.

Levelling is of great use in draining marshes, making rivers navigable, and bringing waters to a town from some river or spring: In this case it has been found, that if the water has a descent of about 5 inches in a mile, it is sufficient; and when it is much more, the current becomes so rapid as to destroy the banks of the river.

SECT. IV.

Of MENSURATION.

BY Mensuration is commonly meant the manner of measuring and computing the dimensions of the several parts of a fortification, or of any other building both in civil and military architecture: Every country has its particular measures; here in *England* the foot, yard, fathom, or pole,

pole, are the meaſures generally uſed, which being well known, need no farther explanation.

Menſuration is either lineal, plane, or ſolid; the lineal is uſed in ſurveying and levelling; the plane, in finding the contents of fields, floors, paintings, glaſſes, and, in general, of every thing where the ſuperficies are only required; and the ſolid, in meaſuring walls, timbers, excavations, *&c.*

It has been proved * in geometry, that paral- * *art.* 40.
lelograms or rectangles which have equal baſes and equal altitudes, are equal: Hence ariſes the common rule, that the area of a rectangle or ſquare is found by multiplying their ſides together. It has alſo been * ſhewn that a triangle is * *art.* 35.
equal to half the rectangle of the ſame baſe and altitude: And as all rectilinear figures may be reduced into triangles, it is evident that their areas or contents may be found by the ſame rule.

Though the reaſon of this rule is obvious, yet why one length multiplied by another ſhould produce ſquare ſuperficies, has either been much neglected, or unknown to moſt authors and teachers of arithmetick, who generally propoſe the erroneous queſtion of multiplying money by money.

In order to make this plain, ſuppoſe the ſides *Plate* XIII.
of a ſquare A B C D to be divided into any num- *Fig.* 4.
ber of equal parts, ſuch as A E, by lines parallel to the ſides; then, whatever the number of parts are into which A D is divided, it is evident, that the rectangle A B F E, whoſe baſe A E is one of them, will contain juſt as many ſmall ſquares; that is, if A D be divided into ſix equal parts, the rectangle A F will contain ſix ſmall ſquares: It is no leſs evident, that the ſquare A B C D

tains alſo as many rectangles, ſuch as AF, as the ſides are divided into equal parts: Therefore, if the number of ſquares contained in the rectangle AF, be multiplied by the number of the rectangles contained in the great ſquare ABCD, the product will expreſs the number of ſmall ſquares contained in the great one AC.

If ABCD be a rectangle, and the ſide AD is to the ſide AB, as to 10 to 12; it is evident, that if the baſe AE be the tenth part of AD, the rectangle AF will contain 12 ſmall ſquares, whoſe ſides are equal to AE; and the figure ABCD will contain ten rectangles ſuch as AF. Conſequently, if the number 12 of the ſquares contained in AF be multiplyed by the number 10 of the rectangles, the product 120 will expreſs the number of ſmall ſquares contained in the rectangle ABCD.

If the baſe of a triangle be 8, and the perpendicular 12, then the rectangle of the ſame baſe and altitude will contain 8 times 12, or 96 ſmall ſquares, whoſe ſides are unity; and as the triangle is half this rectangle, half of 96, that is 48, will expreſs the number of the ſquares contained in the triangle.

Hence, a ſquare foot contains 12 times 12, or, 144 ſquare inches; a ſquare yard 3 times 3, or 9 ſquare feet; and a ſquare fathom 6 times 6, or 36 ſquare feet: Therefore, if the area of any figure be expreſſed in ſquare inches, by dividing it by 144, the number of ſquare inches in a foot, the quotient will expreſs that area in ſquare feet; if the area be expreſſed in ſquare feet, by dividing it by 9 or 36, the quotient will expreſs the number of ſquare yards or fathoms contained in that area.

This

This method of finding the areas of figures is eaſy, when their dimenſions are expreſſed by numbers of the ſame denomination; but when they are expreſſed by feet, inches, and decimal parts of an inch, they muſt be reduced into the loweſt denomination; and afterwards, the product divided by the number of ſquare inches in a foot, yard, or fathom, according as the area is to be expreſſed in feet, yards, or fathoms; by which the operations become very tedious and troubleſome.

It is on this account that the following method is uſed, which is both ſhorter and eaſier; and is, that the parts of a lower denomination are adapted to that of a higher, in the ſame manner as in linear menſuration: That is, 12 inches make always a foot, 3 feet a yard, or 6 feet a fathom. Which ſhews, that all the parts of the ſame ſuperficies are reduced to the ſame height; ſo that a ſuperficial foot, referred to yards, is a rectangle whoſe baſe is a foot and height a yard; an inch referred to yards, a rectangle whoſe baſe is an inch and height a yard; if an inch or foot be referred to fathoms, their heights are a fathom; and ſo on. Conſequently, a foot or inch multiplied by a yard, gives a ſuperficial foot or inch of a yard; or if a foot or inch be multiplied by a fathom, the product will be a ſuperficial foot or inch of a fathom.

The ſame thing is to be underſtood in regard to ſolid contents; for if the cube AB be divided into any number of equal parts, ſuch as ADC by planes, parallel to one of the ſides AC, and ADC be the ſixth part of a cubic fathom, it is called a ſolid foot of a fathom; therefore a ſolid foot of a fathom has a ſquare fathom for its baſe, and a foot in height; in the ſame manner a *Fig.* 5.

ſolid foot or inch of a cubic yard has a ſquare yard for its baſe, and a foot or inch in height. Conſequently, a ſquare yard multiplied by a foot or an inch, gives a ſolid foot or inch of a yard; and a ſquare fathom, multiplied by a foot or an inch, gives a ſolid foot or inch of a fathom.

EXAMPLE I.

414. *A floor being 6 yards 2 feet 6 inches long, and 5 yards 2 feet 4 inches broad, what is the content of the floor expreſſed in yards?*

Say 5 times 6 inches gives 30 inches, or 2 feet 6 inches, ſet down 6 inches and carry 2; then 5 times 2 feet gives 10 feet and 2 carried is 12, or four yards; ſet down 0 and carry 4; and 5 times 6 gives 30, and 4 carried, is 34 yards.

6	2	6
5	2	4
34	0	6
4	1	8
	2	$3\frac{1}{3}$
39	1	$5\frac{1}{3}$

Now becauſe 2 feet are two thirds of a yard, begin with the yards and ſay, the two thirds of 6 yards are 4, ſet down 4 yards; the two thirds of 2 feet are 1 foot 4 inches, ſet down 1 foot and carry 4; and the two thirds of 6 inches are 4, and 4 carried make 8, ſet down 8 inches.

As 4 inches are the ſixth part of 2 feet, or 24 inches, we are to take the ſixth part of the laſt product for 4 inches, by ſaying, the ſixth part of 4 yards or 12 feet is 2 feet, the ſixth part of a foot, or 12 inches, is 2 inches; and the ſixth part of 8 inches is $1\frac{1}{3}$, which, with the 2 carried, makes $3\frac{1}{3}$ inches.

Now all theſe products being added, by carrying

ing 1 foot for 12 inches, and 1 yard for 3 feet, we shall have 39 yards 1 foot and $5\frac{1}{3}$ inches for the content of the floor.

DEMONSTRATION.

We have shewn that yards multiplied by feet and inches gives superficial feet and inches of a yard; and because 1 yard multiplied by 6 yards 2 feet 6 inches, gives the same number of yards, feet, and inches, it is evident, that 2 feet, which are two thirds of a yard, will give also the two thirds of these numbers; and by the same reason 4 inches, or one sixth part of 2 feet, will give the sixth part of the product of 2 feet.

EXAMPLE II.

415. *If a piece of painting is* 12 *yards* 1 *foot* 9 *inches long, and* 7 *yards* 1 *foot* 4 *inches broad, how many yards does this superficies contain?*

Say 7 times 9 inches is 63 inches, or 5 feet 3 inches, set down 3 inches and carry 5; 7 times 1 foot is 7, and 5 carried is 12, set down 0, and carry 4 yards; 7 times 2 is 14, and 4 carried is 18, set down 8 and carry 1; 7 times 1 is 7, and 1 carried is 8.

12	1	9
7	1	4
88	0	3
4	0	7
1	1	$2\frac{1}{3}$
93	2	$0\frac{1}{3}$

As 1 foot is a third part of a yard, say one third of 12 yards is 4, set down 4 yards; one third of 1 foot, or 12 inches, is 4, set down 0 and carry 4; one third of 9 is 3, and 4 carried is 7, set down 7 inches.

And because 4 inches are one third of 1 foot, or 12 inches, take one third of the last product, by saying one third of 4 yards is 1 yard and 4 feet,

feet, ſet down 1 yard, and carry 1 foot to the feet; and one third of ſeven inches is $2\frac{1}{3}$ inches, ſet down this to the inches; add all theſe products together, and there will be 93 yards 2 feet and $\frac{1}{3}$ of an inch.

N. B. It muſt be obſerved that floors, wainſcotting, painting, and window glaſſes, are all meaſured by ſquare yards; and their dimenſions are taken by applying a thread cloſe to all the moldings; and the ſides of a room are taken as if there were no vacancies; and the openings of the windows are taken ſeperately and ſubtracted from the firſt ſum, in order to have the true content of the work.

EXAMPLE III.

416. *Let a Wall be* 50 *fathoms* 3 *feet long, and* 3 *fathoms* 4 *feet* 6 *inches high; how many ſquare fathoms does this wall contain?*

Say 3 times 3 feet is 9 feet, or 1 fathom and 3 feet, ſet down 3 feet, and carry 1 to the fathoms; then as 4 feet are the two thirds of a fathom, and two thirds are not eaſily taken in large numbers, I take one third for 2 feet, and ſet the product down twice; and ſince 6 inches are one fourth of 2 feet, or 24 inches, take one fourth of the product of 2 feet, and the ſum of all theſe products, gives 189 fathoms 2 feet and 3 inches for the content required.

50	3	0
3	4	6
151	3	0
16	5	0
16	5	0
4	1	3
189	2	3

EXAMPLE IV.

417. *If an area is* 24 *fathoms* 2 *feet* 9 *inches one way, and* 7 *fathoms* 9 *inches another; how much is the content expreſſed in fathoms?* Here

Here I multiply the upper factor by 7 fathoms, in the usual manner; then as there are no feet in the multiplier, I suppose 2 feet, which gives 8 fathoms and 11 inches; this product must be crossed, as not belonging to the area; then I take for 6 inches the fourth part of this product, and for 3 inches half the product of 6; and the sum, exclusive of the product of 2 feet, gives 174 fathoms, 1 foot, and $6\frac{3}{8}$ inches, for the content of the area required.

	24	2	9
	7	0	9
	171	1	3
	ß	ø	ɪɪ
	2	0	$2\frac{1}{4}$
	1	0	$1\frac{1}{8}$
	174	1	$6\frac{3}{8}$

EXXMPLE V.

418. *Let the dimensions of a solid be such as expressed in the margin, to find its contents.*

Here you multiply any two dimensions together, as before; suppose the second and third which gives 85 yards, 2 feet, 11 inches, and this sum is again multiplied by the first dimension 20 yards, 2 feet, 6 inches, in the usual manner, and the product will be as in the margin.

It must be observed, that instead of taking the two thirds for 2 feet, we have taken one third for one foot, and set the product down twice, as being easier in large numbers; we have also neglected the fractions of inches, as being of no manner of consequence.

Yds.		
20	2	6
12	1	4
6	2	9
74	2	0
8	0	10
2	0	1
1	0	0
85	2	11
20	2	6
1719	1	4
28	1	11
28	1	11
14	0	11
1791	0	1

EX-

EXAMPLE VI.

419. *Let there be an excavation of earth, whose dimensions are expressed in the margin; to find its solid content.*

	Fath.		
	10	1	7
	25	2	8
	6	1	4
	256	3	7
	3	2	6
	1	0	10
	261	0	11
	6	1	4
	1566	5	6
	43	3	1
	14	3	0
	1624	5	7

Here the first denomination is fathoms, and the first and second dimensions are multiplied together, then the product by the third.

It must be noted when 1 of the multiplier is of more than one figure,

It is more expeditious to multiply them separately, and then reduce them to a higher dimension; as here 25 fathoms multiplied by 7 inches gives 175 inches, which being divided by 12 gives 14 feet and 7 inches; the inches are set down, and the feet carried to the place of feet in the usual manner.

In the application of mensuration to masonry, it is the custom to reduce the thickness of the wall to a brick and a half; for which reason, having found the superficial contents, say as 3 is to the thickness of the wall expressed in half bricks, so is the superficial content of the wall, reduced to a brick and a half thick.

SECT.

SECT. V.

METHOD *for measuring* SURFACES *and* SOLIDS, *by means of their centers of gravity.*

DEFINITION.

THERE is a point in all bodies; such, that if they are supported or suspended thereby, all their parts will remain at rest; this point is called the Center of Gravity.

Though lines and surfaces have no weight, yet as the consideration of their centers of gravity is of great use, we shall consider their contents in the the same manner as in the solids, as so many weights proportional to them.

There arises a remarkable property from the center of gravity; which is, *that the content of a surface or solid, described by the parallel motion of a line or surface, is always equal to the product of the generating quantity multiplied by the perpendicular distance of its center of gravity, from the beginning of its motion; and the content of a surface or solid, described by a circular motion, is always equal to the product of the generating quantity multiplied by the circular arc described by the center of gravity.*

This property has been demonstrated in my treatise of conic-sections, page 147; and as it depends on the method of fluxion and fluents, which we don't propose to treat of here, we shall give

give only as many examples as are neceſſary for our purpoſe.

COR. I.

420. Hence, if any two of the three things which enter into the computation, *viz.* the generating quantity, its center of gravity, or the quantity generated, are given, the third will likewiſe be given.

COR. II.

421. It is evident that the center of gravity of a right line is the point which biſects it; for if it be ſupported or ſuſpended by that point, there is no reaſon why it ſhould incline more one way than another.

THEOREM I.

Fig. 6. 422. *The center of gravity of a rectangle* ABCD, *is the point* L *which biſects the diagonal* BD.

For, if the line ELH be drawn parallel to the ſide AB or DC, it will biſect the rectangle; and therefore the center of gravity is ſomewhere in that line; and if there be imagined another line, drawn through the point L, parallel to the ſide AD or BC, this line will alſo biſect the rectangle, and ſo the center of gravity will likewiſe be ſomewhere in that line. Conſequently, the common interſection L, of theſe two lines, will be the center of gravity of the rectangle.

THEOREM II.

423. *If any two ſides* AB, AD, *of a triangle* ABD, *are biſected by lines* DP, BE, *drawn from the*

the oppoſite angles, their common interſection O *will be the center of gravity of the triangle.*

For ſince the line BE biſects the ſide AD, it will biſect * the triangle ABD, as likewiſe all its parts cut off by lines * drawn parallel to that line; and therefore the center of gravity of the triangle is ſomewhere in that line: And becauſe the line DP biſects AB, it will likewiſe biſect that triangle and all its parts cut off by lines drawn parallel to AB; and ſo the center of gravity is alſo ſomewhere in this line. Conſequently, the common interſection O of the lines DP and BE, will be the center of gravity of the triangle.

* art. 42. * art. 67.

COR. I.

424. Hence, if from the point E the line En be drawn parallel to DP; then becauſe AE and ED are equal by ſuppoſition, An will be * equal to nP; and ſince AP is always equal to PB by ſuppoſition, nP will be one third of nB; and therefore EO will likewiſe be one third of EB; ſince * BP: Pn:: BO: OE.

* art. 67. * art. 67.

COR. II.

425. If from the center of gravity O, of the triangle ABD, the lines OF, OG, are drawn parallel to the ſides AB, AD; then becauſe BO: BE:: GO or AF: AE; and as BO is two thirds of BE; AF will likewiſe be two thirds of AE; But two thirds of half is but one third of the whole; and therefore AF is one third of the baſe AD, and FD two thirds of that baſe. By the ſame reaſon AG is one third of the ſide AB, and GB the other two thirds.

The two preceding theorems may alſo be demonſtrated thus,

The

The cylinder described by the rotation of the rectangle ABCD, about one of its sides AB as an axis, has been proved to be equal * to the product of its base and altitude; if therefore $AB = b$, $AD = a$, the circumference of the radius AD, c; then will $\frac{ac}{2}$ be the area of the base, and $\frac{abc}{2}$ the content of the cylinder, which being divided by the generating quantity ABCD (ab) leaves $\frac{c}{2}$ for the circumference described by the center of gravity. But as the distance AE of the center of gravity L of the rectangle AC is half the side AD, the circumference described by the center of gravity L will be $\frac{c}{2}$ half that described by the radius AD.

*art. 235.

It has also been proved * that $\frac{abc}{6}$ expresses the content of the cone described by the rotation of the right angled triangle ABD, about the side AB as an axis; and as this triangle is equal to $\frac{ab}{2}$; dividing the content $\frac{abc}{6}$ of the cone by the generating quantity, we get $\frac{c}{3}$ for the circumference described by the center of gravity O of that triangle, whose radius AF is therefore one third of AD.

*art. 239.

If the centers of gravity were of no other use than to find only the surfaces and solids in a different manner from that given before, it would seem to be superfluous to trouble the reader with repetitions; but the great use of these centers consists in finding parts of solids, such as of a hollow cylinder,

linder, fruſtrum of a cone, and ſome others of that nature, which are to be known in the menſuration of vaults, arches, orillons, and circular flanks, in a much eaſier and conciſer manner, as will be ſeen hereafter.

EXAMPLE I.

426. *To find the ſolid content of the hollow cylinder deſcribed by the rectangle* EC *about the axis* AB. Fig. 6

Let AE be 8, ED 6, and DC 10; then the rectangle EC will be 60; and as the diſtance from the center of gravity of the rectangle EC to the axis of rotation AB, is equal to AE (8) and to half of ED (6) or to 11; and the diameter of a circle is to its circumference * as 7 is to 22; if we ſay $7:22::22:69\frac{1}{7}$; this fourth term will be the circumference deſcribed by the center of gravity of the rectangle EC. Therefore, the product of 60, the generating plane, multiplied by $69\frac{1}{7}$, the circumference deſcribed by its center of gravity, will give $4148\frac{4}{7}$ for the content required. * *art.* 252.

EXAMPLE II.

427. *To find the content of the ſolid deſcribed by the triangle* EHD *about the axis* AB.

The ſame thing being ſuppoſed as before, the triangle EHD will be 30, as being half the * rectangle EC; and the diſtance from its center of gravity to the axis of rotation AB, equal to AE (8) and to * one third of ED (6) or to 10. If therefore $7:22::20:62\frac{4}{7}$; this fourth term will be the circumference deſcribed by the center of gravity of the triangle. Conſequently, $62\frac{4}{7}$ multiplied * *art.* 35. * *art.* 425.

tiplied by 30, gives $1885\frac{5}{7}$ for the content required.

EXAMPLE III.

Fig. 7. 428. *To find the content of a right cylinder* P, *from which a fruſtrum of a cone has been ſubtracted.*

Let ABCD be the ſection of this ſolid, or the generating plane; GL the axis of rotation; draw BH parallel to CD; then the ſolids deſcribed by the triangle ABH, and the rectangle HC, muſt be found ſeparately, and their ſum will be the content required.

If AG be 4, AH, 6, HD, 8, and DC, 12, then the triangle ABH will be half of 6×12 or 36; and the diſtance from its center of gravity to the axis of rotation GL is equal to AG (4) and to two thirds * of AH (6) or 8; and if $7 : 22 :: 16 : 50\frac{2}{7}$, this fourth term will be the circumference deſcribed by the center of gravity. Therefore 36, multiplied by $50\frac{2}{7}$, gives $1810\frac{4}{7}$ for the content of this ſolid.

* *art.* 425

The rectangle HC is equal to 8×12, or 96, and the diſtance from its center of gravity to the axis GL of rotation, equal to GH (10), and to half of HD (8) or 14; and if $7 : 22 :: 28 : 88$, this fourth term multiplied by 96 gives 8448 for the content of the ſolid. Conſequently, the ſum of $1810\frac{4}{7}$ and 8448 gives $10261\frac{4}{7}$ for the content required.

EXAMPLE IV.

429. *To find the content of a fruſtrum of a right cone* Q, *from which a cylinder has been ſubtracted.*

Let ABCD be a ſection through the axis of this ſolid, EF the axis of rotation: If ED be 3, and

and the reſt as before, then the center of gravity of the rectangle from the axis EF is equal to ED(3), and to half of DH(8) or 7; therefore, the circumference deſcribed by this center of gravity is 44, and the rectangle 96, multiplied by 44, gives 4224 for the content of the ſolid deſcribed by that rectangle. The diſtance from the center of gravity of the triangle ABH, to the axis EF of rotation, is equal to EH(11), and to one third of AH(6) or 13; and if $7 : 22 :: 26 : 81\frac{5}{7}$, then $81\frac{5}{7}$, multiplied by the triangle 36, gives $2941\frac{5}{7}$ for the content of the ſolid. Conſequently, the ſum of $2941\frac{5}{7}$ and 4224 gives $7165\frac{5}{7}$ for the content required.

Theſe two laſt examples are particularly uſeful in the meaſuring of the orillons and retired flanks.

EXAMPLE V.

430. *To find the content of the ſolid deſcribed by the rotation of the ſemi-circle* ABC, *about the tangent* AT, *perpendicular to the diameter* AC. Fig. 8.

It is evident that the center of gravity of the ſemi-circle is ſomewhere in the radius OB, parallel to AT, ſince that radius biſects all the lines that can be drawn parallel to the diameter AC. If therefore the radius AO is 14, the circumference of that radius will be 88, and the area ABC will be 14×22, or 308, which being multiplied by 88, gives 27104 for the content required.

Whether the axis AT of rotation touches the circle, or is at ſome diſtance from it, the content of the ſolid deſcribed about that axis is always found in the ſame manner; provided the circumference deſcribed by the center of gravity

be rightly found, that is, according to its distance from the axis of rotation.

But if the solid described by the semi-circle about the tangent TB, parallel to the diameter AC, be required, the distance of the center of gravity from that line must be known.

EXAMPLE VI.

431. *To find the center of gravity* L *of the semi-circle.*

If the radius AO be a, the semi-circumference c; then the area of the circle * will be ac, and the solid content of the sphere * $\frac{4aac}{3}$: Now as this solid is likewise equal * to the product of the generating quantity ABC, multiplied by the circumference described by the center of gravity L, if the solid $\frac{4aac}{3}$ be divided by the area $\frac{ac}{2}$ of the semi-circle, we shall have $\frac{2a}{3}$ for the circumference described by the center of gravity; and if $2c : a :: \frac{2a}{3} : \frac{aa}{3c}$, this fourth term will be the distance required from the center O of the circle.

* *art.* 173.

* *art.* 216.

* *art.* 420.

EXAMPLE VII.

432. *To find the content of the solid described by the semi-circle about the tangent* TB, *parallel to the diameter* AC.

The same thing being supposed as before, if OL $(\frac{aa}{3c})$ be subtracted from OB (a) we shall have

have $a - \frac{aa}{3c}$ equal to LB; and if $a : 2c :: a - \frac{aa}{3c} : 2c - \frac{2a}{3}$; this fourth term will be the circumference described by the center of gravity about the axis TB; which being multiplied by the area $\frac{ac}{2}$ of the semi-circle, gives $acc - \frac{aac}{3}$, or $ac \times \overline{c - \frac{1}{3}a}$ for the content of the solid required.

If the radius AO *(a)* be 7, then the semi-circumference *c* will be 22; and therefore $7 \times 22 \times 19\frac{2}{3}$ or $3028\frac{2}{3}$ will be the content of the solid.

This last example is useful in finding the solid content of a vault, called by the *French*, Bonnet de Pretre.

What has been said here is sufficient to shew the manner of applying the center of gravity to the finding the contents of surfaces and solids, in a most compendious and concise manner; some part of its use will appear in what follows.

MENSURATION *applied to* FORTIFICATION.

When the plan of a fortification is traced, there *Plate* is a line which is called the Principal, or Majis- XIV. tral, on which the length of the parts are set off; *Fig.* 10. and it is this line which is represented by the cordon, marked by the letter C in the profil: For instance, when it is said that the faces of a bastion are 50 toises, it is to be understood near the cordon, where the slope of the wall begins.

In order to measure the revetement of the face *Fig.* 12. of the bastion AD, we must consider the profil (fig. 10) ABCD, which we shall suppose, according

cording to Mr *Vauban*, to be 30 feet, or 5 toiſes high from the foundation AD to the cordon BC, 5 feet at BC, and 11 at AD; adding theſe numbers together we get 16 feet for their ſum, half of which, 8 feet, or 1 toiſe and 2 feet, being multiplied by the height AB(5) gives 6 toiſes 3 feet ſquare for the area ABCD.

The foundation ADIH muſt be meaſured by itſelf, and cannot be determined here, as depending on the nature of the ground, but being always rectangular may be computed eaſily; the upper part of the wall EFGC muſt likewiſe be meaſured ſeparately.

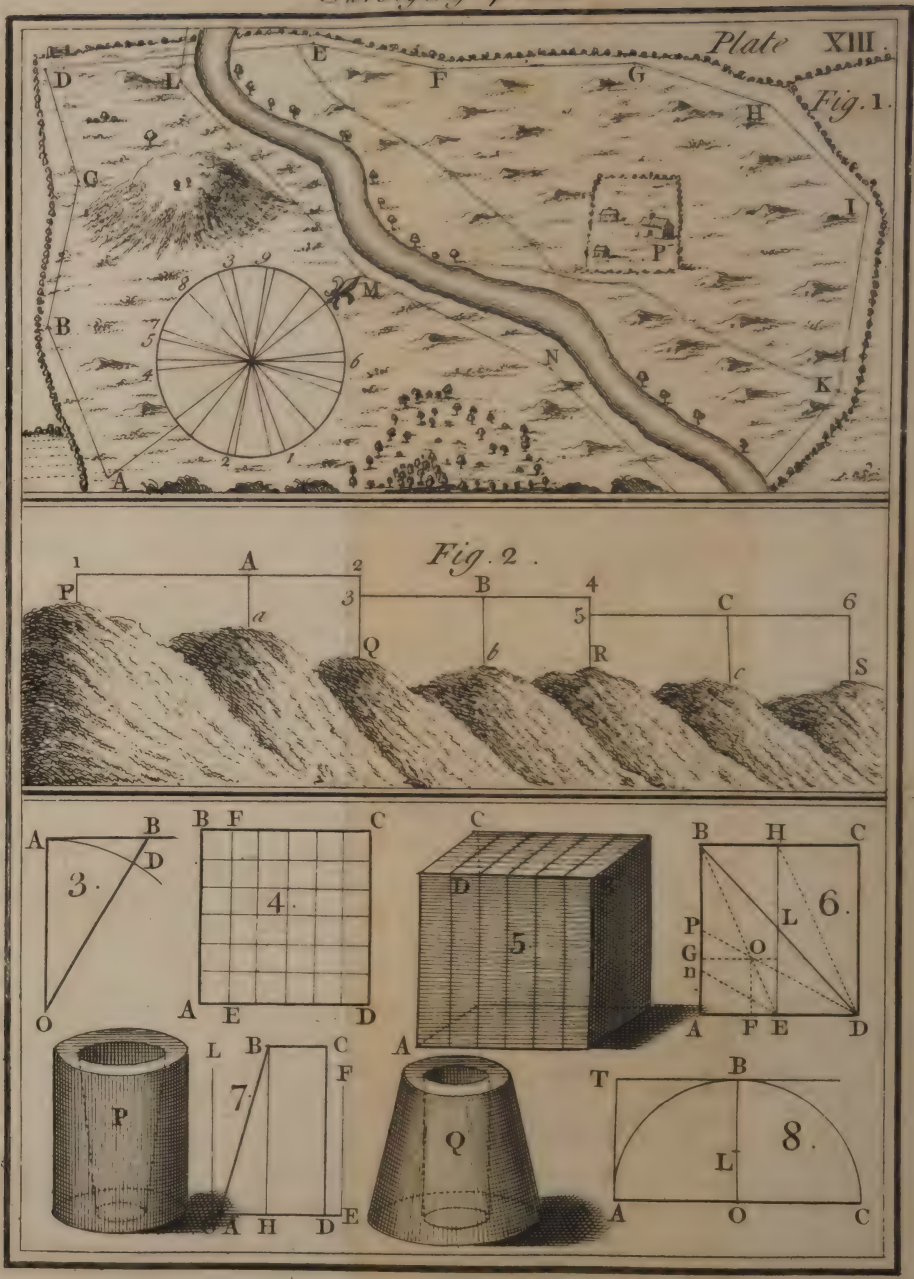
Now ſuppoſe the twelfth figure repreſents the wall from the foundation to the cordon; from the inner angle A draw AB, AC, perpendicular to the faces, through which ſuppoſe to paſs two vertical planes; then the angular part cut off will be a fruſtrum of a pyramid, like the fig. 11; where AB and AD are each 5 feet, the angles at B and D right ones and the angle C equal to that of the ſaliant angle of the baſtion, which has been * found to be 83 degrees and 8 minutes in the exagon; and ſince the ſides CB, CD, are equal, they will be found by trigonometry to be 5.6 feet.

Fig. 11.

**art.* 395

The angle L at the baſe being equal to the angle C, that is, of 83 degrees and 8 minutes, the ſide EF, 11 feet as well as its equal EN, the ſide FL and its oppoſite one will be found to be, by the ſimilarity of the triangles ACD, EFL, 12.32 feet. Therefore, having the height, the lower and upper baſe of this fruſtrum, its ſolid content is found to be 31.26 toiſes. And ſubtracting the ſide CD(5.6) feet, fig. 11, from the face of the baſtion, fig. 12, or from 50 toiſes, we ſhall

Plate XIII.



ſhall have 49 toiſes 4 tenths of an inch for the part AD of the face of the baſtion to the orillon: This length multiplied by the content 6 toiſes 3 feet of the profil, gives 318 cubic toiſes and 5.6 feet, for the content of the face AD.

We have found * the radius with which the orillon is deſcribed to be 5.13 toiſes, the diſtance from the center of gravity of the rectangular part ABCL, fig. 10, of the profil, whoſe baſe AL being 5 feet, to the center of that arc, or to the axis of rotation, will be * 27.4 feet, and the diſtance from the center of gravity of the triangle LCD is 32 feet: So that having the arc of * 11.16 toiſes, which the orillon makes near the cordon, the arc deſcribed by the center of gravity of the rectangle ABCL will be 10 toiſes nearly, and that deſcribed by the center of gravity of the triangle LCD is 11.6 toiſes; and becauſe the rectangle ABCL is 4.16 toiſes, the part of the orillon deſcribed by that rectangle will be 41.6 toiſes, and triangle LCD being 5 toiſes, the part of the orillon deſcribed by that triangle will be 58 toiſes. Conſequently, the orillon contains 99.6 cubic fathoms or toiſes.

*art. 401.

*art. 422.

*art. 401.

The ſtrait part FE, fig. 12, of the wall between the orillon and retired flank, being 5 toiſes long by conſtruction, is eaſily found, for which reaſon we ſhall omit it here.

As to the retired flank EL, its radius has been found to be * 18.86 toiſes, and the arc 19.75 toiſes; the radius of the arc deſcribed by the center of gravity of the rectangle ABCL, fig. 10, will be 19.7 toiſes, and the arc deſcribed by that center 20.6 toiſes nearly; the radius of the arc deſcribed by the center of gravity of the triangle LCD is 18.53 toiſes; and the arc deſcribed by

*art. 400.

that center 19.98 or 20 toiſes nearly. Therefore, the part of the retired flank deſcribed by the rectangle ABCL(4.16) will be 85.69 toiſes; and the part deſcribed by the triangle LCD(5) is 100 toiſes. Conſequently, the retired flank contains 185.69 toiſes.

The broken part LH, fig. 12, of the curtain, and the curtain itſelf, being ſtrait walls, are eaſily found, by multiplying the profil by their lengths.

It muſt be obſerved that in all the ſaliant angles the parts cut off by planes perpendicular to the walls from the inner angles are always fruſtrums of pyramids, like that repreſented by the 11th fig. whoſe upper and lower baſes are known from the given dimenſions of the profil; and the parts cut off in the re-entring angles are ſuch fruſtrums as are repreſented by fig. 13, from which there is cut off a ſmall pyramid, ariſing from the ſlope of the wall; and as theſe parts of the wall are always regular ſolids, their contents may be found in the manner ſhewn before; the ſame thing may be ſaid in regard to the counterforts, which are regular priſms.

What has been ſaid in reſpect to the body of the place, may likewiſe be applied to all kinds of outworks; the round parts of the counterſcarp are found in the ſame manner as the retired flanks, for which reaſon we ſhall inſiſt no longer on this ſubject.

MENSURATION *applied to* VAULTS *and* ARCHES.

There are three ſorts of arches, the circular, the elliptic or oval, and the gothic; we ſhall begin

Plate XIV.

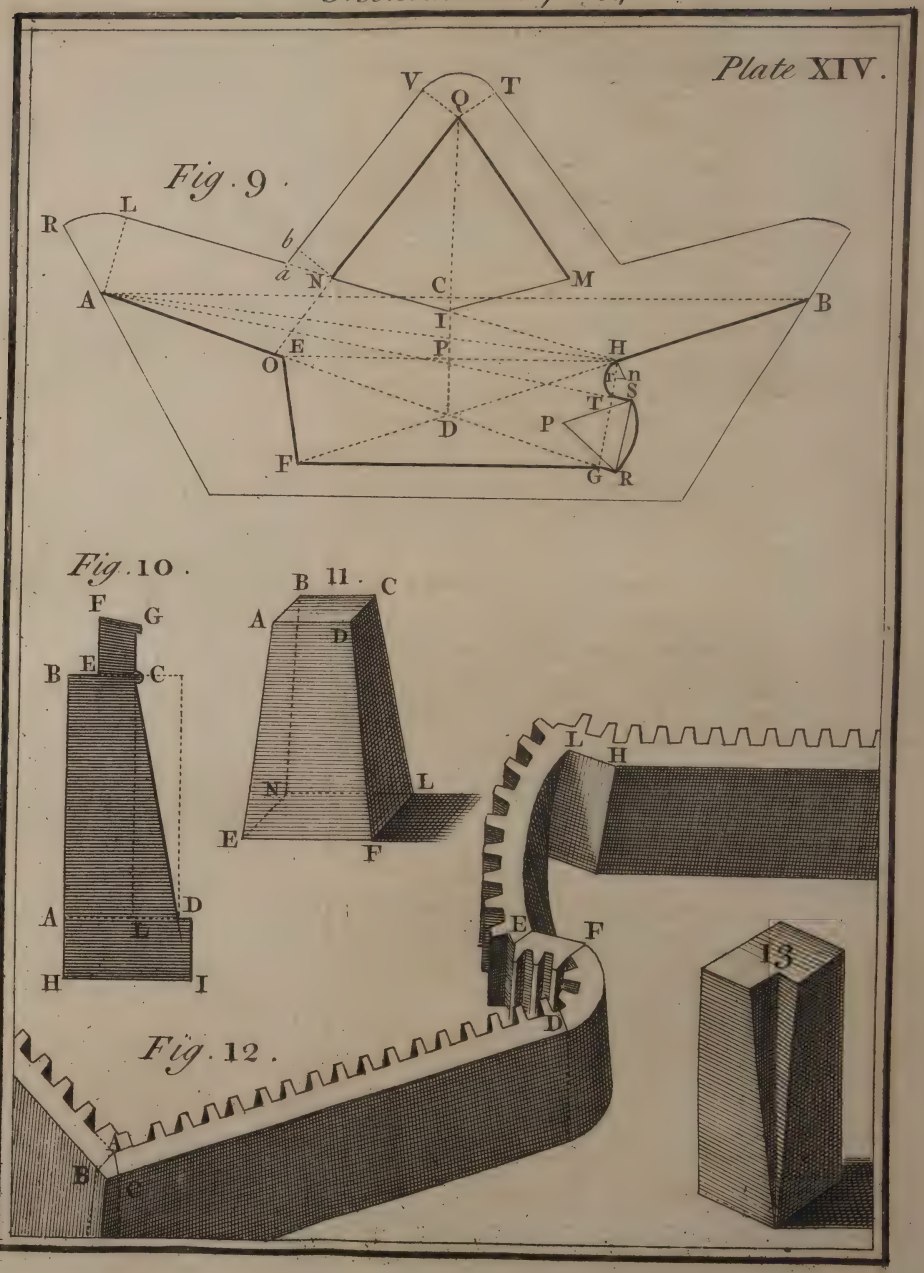


Fig. 9.
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Fig. 10.
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B 11. C
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Fig. 12.
A
B

gin with the menſuration of the firſt, as being moſtly uſed, eſpecially here in *England*.

The firſt thing of this kind to be conſidered in a fortification are the powder magazines; their planes are generally as is repreſented by fig. 14, *Plate* XV. and the ſection as fig. 15; the outſides of the roofs *Fig.* 14, being always in the form of a right angled tri- 15. angle

The common method for meaſuring theſe arches is to conſider the part EDF as a ſolid pyramid, whoſe baſe is the triangle EDF, and whoſe altitude the length of the magazine; then the content of the ſemi-cylinder, whoſe baſe is the ſemi-circle APB, and the axis is the ſame as that of the pyramid, being ſubtracted from the pyramid, the difference gives the content of the upper part of the magazine, cut off by the horizontal line EF. But the work will be ſhorter if the content of the ſemi-circle APB be ſubtracted from that of the triangle EDF, and the difference multiplied by the length of the inſide AC, fig. 14, of the magazine.

The ſide walls are meaſured by multiplying the *Fig.* 14. plane or baſe ABCD by the height LA, fig. 15, of the peers; as to the foundations and the two gable ends, their contents are eaſily found, as being regular ſolids.

The ſally ports being likewiſe arches much like the preceding one, only the outſide being circular and concentric with the inſide, inſtead of triangular; ſo there is no more to do than to ſubtract the content of the inner ſemi-circle from that of the outward one, and multiplying the difference by the length of the arch; and finding the ſide walls as before.

If

If the arches are elliptical, such as most gateways generally are, then because the area of the circle described with the greater axis as diameter is to the area of the ellipsis, as * the greater axis is to the lesser. These kind of arches may be measured with as much ease as if they were circular.

*art. 316.

In order to measure the gothic arches, it is necessary to know their construction: The opening AB of the arch is divided into three equal parts AO, OC, and CB; from the point C, as center, with CA the two thirds of AB as radius, the arc AL is described; and from the other point O as center, with the same radius as before, the arc BL is described.

Fig. 16.

Hence, if the line DLP bisects AB at right angles, CP being the half of one third of AB will be the sixth part of AB; and as CA or CL is two thirds of the same line, if we say as $\frac{2}{3}$ is to $\frac{1}{6}$, or as 4 is to unity, so is the radius 100000 to the cosine 25000 of the angle LCP; this angle will be 75 degrees 31 minutes; the perpendicular PL is likewise found to be .065 parts of AB: Now if the content of the sector CAL be found, and the area of the triangle CPL subtracted, the difference will give the area ALP of half the inside arch; so that if twice that difference be subtracted from the area of the triangle EDF, and the remainder multiplied by the length of the arch, you will have its solid content.

The most difficult part of measuring arches is when there are several that cross each other at right angles, and are supported by pillars, such as those under the treasury near St *James*'s park; and above all, when they are made in the gothic manner,

manner, as thoſe at *Weſtminſter:* But as there are but few or none built now in that manner, we ſhall only inſiſt on the method of meaſuring the former.

The 17th figure repreſents the manner in which they are joined or meet each other; this is nothing elſe than two ſemi-cylinders interſecting each other, ſo that the only difficulty conſiſts in meaſuring the part ABCD. *Fig.* 17.

If the line FG be drawn through the point of interſection E of the arches, parallel to AB or DC, and we ſuppoſe a plane to paſs through that line, perpendicular to the baſe ABCD; it is evident that this plane biſects the fourth part BEC of this arch; and therefore, having found the part BEG or GEC by multiplying it by 8, we ſhall have the content of the whole.

Let AKC be a ſemi-cylinder, whoſe diameter AC is equal to that of the arch; and let ABC be a ſection ſo as its altitude Bn be equal to half the diameter AC; if DBE be a ſection parallel to the baſe AnC, it is eaſy to perceive that the part CBE or ABD is equal to the part BEG, fig. 17, of the arch; and therefore the two parts, CBE, ABD, together; or, which is the ſame, the difference between the ſemi-cylinder ADBEC, and the part ABC cut off, will be equal to the fourth part BEC, fig. 17, of the arch. *Fig.* 18.

If the part ABC be cut by any two parallel planes OnB, Pmr, perpendicular to the baſe AnC of the cylinder, they will form two right angled ſimilar triangles OnB, Pmr: For the lines nB, mr, being parallel and perpendicular to the baſe AnC of the cylinder, by ſuppoſition, will be perpendicular to the parallels On, Pm; and ſince BO, rP, are likewiſe parallel and in the

ſame

ſame plane, they will make equal angles BOn, rPm, with the parallels On, Pm.

* *art.* 88. Now becauſe ſimilar triangles are as * the ſquares of their homologous ſides, the areas OnB, Pnr, are to each other as the ſquares of the lines On, Pm; that is, as the ſquares of the ordinates of a ſemi-circle: But the ſquares of

* *art.* 170 theſe lines are likewiſe as the areas of * the circles which they deſcribe in the rotation of the ſemi-circle AnC about the diameter. Therefore the part ABC will likewiſe be as the ſphere deſcribed by the ſemi-circle AnC. And becauſe the con-

* *art.* 216. tent of the ſphere is found * by multiplying the area of the circle deſcribed by the radius On, by the two thirds of the diameter AC; by a parity of ratios, the ſolid ABC will likewiſe be found, by multiplying the area of the greateſt triangle OnB by the two thirds of the diameter AC.

Hence, becauſe the lines On, nB, are equal by ſuppoſition, and On is the radius of the ſemi-circle AnC, the area of the triangle OnB will be half the ſquare made by the radius; and this multiplied by two thirds of the diameter AC, gives the two thirds of the cube made by the radius, equal to the content of the ſolid ABC.

If the radius OC or OA be a, the ſemi-circumference AnC, c; then will $\frac{2a^3}{3}$ expreſs the

* *art.* 235. content of the ſolid ABC, and * $\frac{aac}{2}$, that of the ſemi-cylinder ADEC: Therefore $\frac{aac}{2} - \frac{2a^3}{3}$ or $\frac{aa}{6} \times \overline{3c - 4a}$ will be the content of their difference ADBEC, or the fourth part of the arch ABCD,

ABCD, fig. 17. Consequently, $\frac{2aa}{3} \times \overline{3c - 4a}$ will express the content of the whole, which gives this

GENERAL RULE.

Subtract twice the diameter of the base from three times the semi-circumference, and multiply the difference by the two thirds of the square made by the radius.

If the diameter be 14 feet, the circumference will be 44; and 28 twice the diameter subtracted from 66 three times the semi-circumference 22, gives 38 for the difference; and as the square of the radius 7 is 49, the two thirds of this is $32\frac{2}{3}$, which being multiplied by 38, gives $1241\frac{1}{3}$ for the content of the solid.

The figure 18, of which we have just now given the content, is the vacant part of these kinds of arches, and therefore is to be subtracted from the whole structure, considered as all solid.

The part or solid ABC is called the Hoof or Ongled, and its content is necessary in the measuring of round towers placed on Batardeaux. *Fig.* 18.

There is a kind of arches called by the *French* Bonnet de Pretre; the ceilings of round towers are sometimes made in this manner, and are supported by a round pillar when the building is spacious.

The vacant part of these arches is a solid described by the rotation of a semi-circle about an axis, which is perpendicular to the diameter, therefore may be measured by what has been said in article 430.

BOOK IV.

Of MOTION.

SECT I.

DEFINITIONS.

1. MOTION is a continual ſucceſſive change of places.

2. Velocity is that affection of motion by which a body moves over more or leſs ſpace in equal times.

3. Uniform motion is when a body moves over equal ſpaces in equal times.

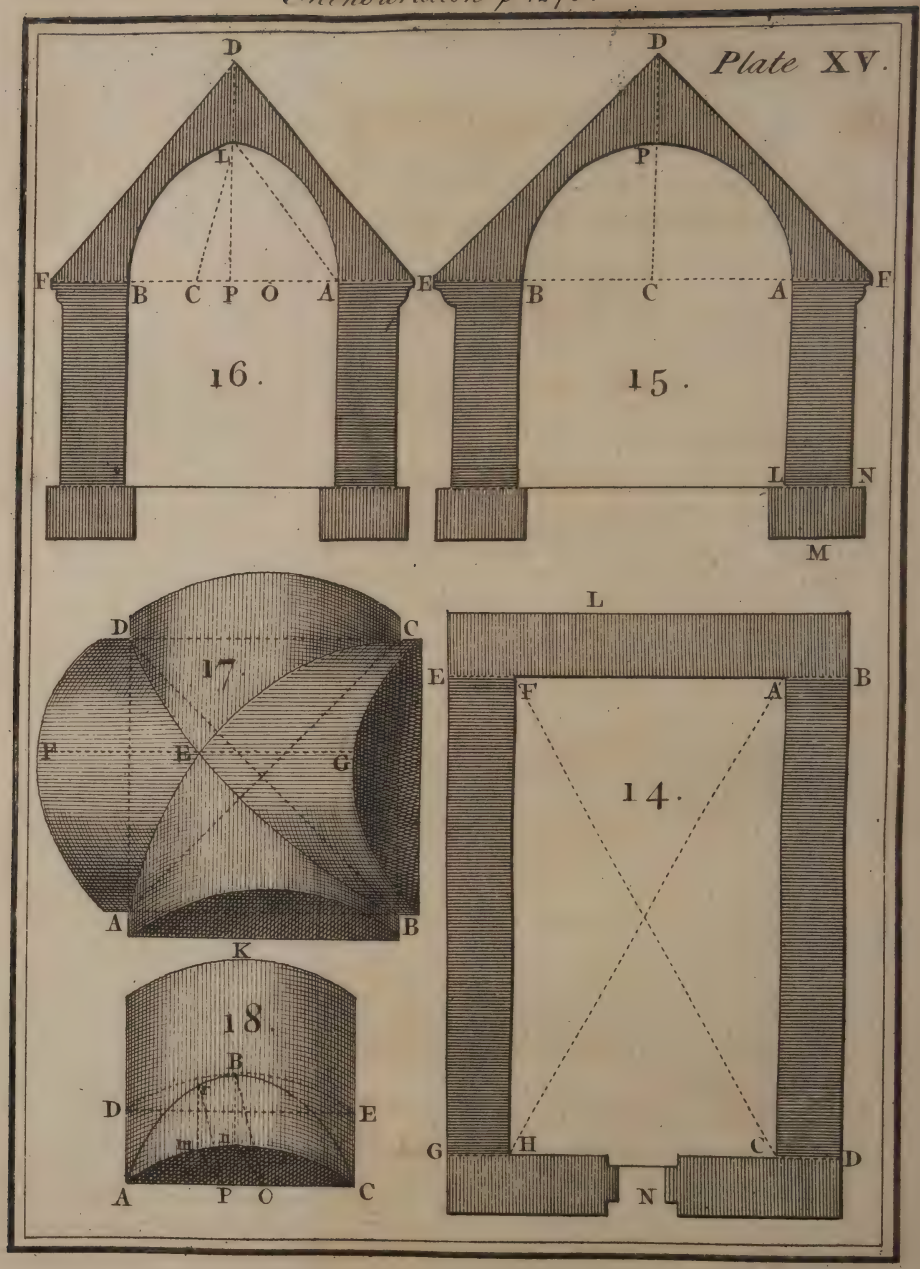
4. Accelerated motion, when the velocity continually increaſes, and retarded when it continually decreaſes.

5. Momentum, or quantity of motion, is the power or force by which a moving body ſtrikes any obſtacle or impediment which oppoſes its motion.

6. Abſolute motion is the change of immoveable places, and relative, the change of moveable places.

It ſeems to be plain that there is no abſolute motion, for all heavenly bodies move continually in their orbits, as far as we know; but let this be as it will, the laws are the ſame in both caſes, as has

Plate XV.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has been confirmed by experiments; for which reaſon we ſhall treat of abſolute motion only.

AXIOM.

Effects are always proportional to their adequate cauſes; that is, a force double produces an effect double, a triple, triple, *&c.*

Of UNIFORM MOTION.

THEOREM I.

433. *Spaces deſcribed with an uniform motion, are as the times in which they are deſcribed.*

For ſince bodies move over equal ſpaces in equal times, by defin. 3, a body will move over twice the ſpace in twice the time, thrice the ſpace in thrice the time, and, in general, n times the ſpace in n times. Therefore, ſpaces moved over with an uniform motion, are as the times in which they are deſcribed.

Thus a man keeping conſtantly the ſame pace, if he walks a mile in one hour, will walk two in two, three in three, and ſo on.

THEOREM II.

434. *Spaces deſcribed with an uniform motion in equal times, are as the velocities with which they are deſcribed.*

It is plain that with a greater velocity a greater ſpace is moved over at the ſame time, and that with a velocity double a ſpace, double will be moved over at the ſame time; with a velocity triple, or quadruple, a ſpace triple, or quadruple; and, in general, with a velocity n tuple, a ſpace n tuple. Conſequently, ſpaces moved over with

an

an uniform motion in equal times, are as the velocities with which they are deſcribed.

THEOREM III.

435. *Spaces uniformly deſcribed are in the compound ratio to that of the times and velocities.*

If two bodies A, B, deſcribe the ſpaces S, s, with the uniform velocities V, v, in the times T, t, reſpectively: I ſay that S:s::TV:tv.

For let another body C deſcribe a ſpace x, in the time T, of the body A, with the velocity v of the body B; then becauſe the bodies B, C, deſcribe the ſpaces s, x with the ſame velocity v, they are * as the times; that is, $x:s::T:t$: And ſince the bodies A, C, deſcribe the ſpaces S, x, at the ſame time they are * as the velocities V, v, *viz.* $S:x::V:v$. Now if the terms of theſe proportions are multiplied together reſpectively, we ſhall have $S\times x:s\times x$, or * $S:s::TV:tv$. Conſequently, ſpaces uniformly deſcribed are in the compound ratio to that of the times and velocities.

*art. 433.
*art. 434
*art. 80.
*art. 95.

COR. I.

436. Hence, if two bodies A, B, deſcribe equal ſpaces with an uniform motion, the velocities are reciprocally as the times. For becauſe $S=s$ by ſuppoſition, we have $TV=tv$ by equality of ratios; and therefore * $T:t::v:V$.

*art. 74.

COR. II.

437. It is evident that the times are as the ſpaces directly, and velocities reciprocally; for if in $S:s::TV:tv$, the antecedents are divided by V, and the conſequents by v, we ſhall * have $S:\frac{s}{v}::T:t$. And, by the ſame manner of rea-

*art. 54.

ſoning

ſoning it may be proved that the velocities are as the ſpaces directly and times reciprocally.

Of COMPOUND MOTION.

THEOREM IV.

438. *Whatever motion, or change of motion, is produced in a body, it muſt be proportional to, and in the direction of the force impreſſed.*

For ſince a body, by reaſon of its inactivity, cannot contribute to the production of its own motion, or of any change therein, it is evident that whatever motion or change of motion is generated in bodies, it muſt entirely proceed from the force impreſſed; and as effects are always proportional to their adequate cauſes, it muſt be proportional thereto, and it muſt likewiſe be directed towards the ſame part with the generating force.

COR.

439. Therefore, if the body whereon the impreſſion is made was in motion before the impulſe, that motion will be accelerated or retarded, according as the force impreſſed conſpires or oppoſes it.

THEOREM V.

440. *If a body be urged at the ſame time by two forces, in the directions of and proportional to the ſides* AB, AC, *of a parallelogram, it will by this compound motion deſcribe the diagonal* AD. Plate XVI. Fig. 1.

For while the force impreſſed in the direction AB cauſes the body to move from A towards B, the force impreſſed in the direction AC will cauſe the body to move from A towards C; but the ſpace moved over in the direction AB is to

the ſpace moved over in the direction AC, as
*art. 438. the force impreſſed in AB to the * force impreſſed in AC, or as AB to AC or BD, by ſuppoſition. If therefore, from any point E in AB, a line EF be drawn parallel to BD, meeting the
* art. 67. diagonal AD in F; then becauſe * AB:BD:: AE:EF; and ſince the ſpace moved over in AB, by the force impreſſed in that direction, is to the ſpace moved over in AC, by the force impreſſed in that direction, as AB to BD: It is evident that if AE repreſents the ſpace moved over in that direction, EF will expreſs the ſpace moved over in the direction AC at the ſame time. Conſequently, the body will be in the diagonal AD; and as this proportion always ſubſiſts, whether AE is great or little, the body has moved in the diagonal AD from the beginning A, and never was out of it.

This propoſition is of the utmoſt importance in mechanics, and in natural philoſophy; for it ſhews how to eſtimate the joint effect of ſeveral powers acting together on the ſame object, and in what manner any given number of powers may be applied, ſo as to produce a deſired effect.

COR. I.

441. Hence, the body moves over the diagonal AD, by the compound motion, in the ſame time that it would move over the ſide AB or AC, by the ſingle force impreſſed in thoſe directions. For we have proved, that while the body moved from A to E by the force impreſſed in that direction, it moved from A to F in the diagonal by the joint forces; and therefore, while the body moved from A to B by the force im-

impreſſed in that direction, it moves from A to D in the diagonal by the joint forces.

COR. II.

442. It is evident, that the force impreſſed in the direction of the diagonal AD, by the joint forces, is to the ſingle force impreſſed in the direction AB, or AC, as AD is to AB or BD: For becauſe the effects are proportional to their adequate cauſes, the force impreſſed in the direction AD, would cauſe the body to move over the diagonal AD in the ſame time that the force in the direction AB would cauſe the body to move over AB; and ſince the joint forces in the directions AB, AC, move likewiſe the body over the diagonal in the ſame time that the force impreſſed in the direction AB would move it over the ſide AB: It follows, that a ſingle force, expreſſed by the diagonal AD, produces the ſame effect as the joint forces expreſſed by the ſides AB, AC, of the parallelogram.

COR. III.

443. It is manifeſt, that as the angle CAB, made by the directions of the two forces, decreaſes, ſo does the effect ariſing from their joint impreſſion increaſe; becauſe the diagonal AD, which expreſſes their effect, increaſes as that angle decreaſes: And therefore, when the angle CAB vaniſhes, or the ſides AC, AB, coincide with the diagonal, the effect will then be the greateſt of all that theſe joint forces can produce.

COR. IV.

444. On the contrary, as the angle CAB, made by the directions of the two forces, increaſes, the effect of their joint endeavours decreaſes; and

when the ſides C A, A B, become one continued right line, the effect of theſe joint forces is the leaſt of all; which gives the two following caſes.

I. If the ſides A B, A C, are equal, the diagonal A D vaniſhes, when theſe ſides become one continued right line; and therefore the two forces being equal, and acting in oppoſite directions, deſtroy each other.

II. If the ſide A C is leſs than the ſide A B, the diagonal A D coincides with the greater ſide A B, when theſe ſides become one continued right line, and is equal to the difference between the ſides A B and B D, or A C. Conſequently, the body will move in the direction A B of the greater force, with a force impreſſed equal to the difference between the forces expreſſed by A B and A C.

COR. V.

Fig. 2. 445. If from the oppoſite angles B, C, the lines B E, C F, are drawn perpendicular to the diagonal A D, the force impreſſed by A B in the direction of the diagonal A D, will be expreſſed by A E, and the force impreſſed by A C in the diagonal by A F. For if the parallelograms E H, F G, are compleated, the forces expreſſed by the
* *art.* 440 ſides A E, A H, or E B, will produce the * ſame effect in the direction A B as the force expreſſed by that line; and the forces expreſſed by the ſides A F, A G, or F C, will likewiſe produce the ſame effect as the force expreſſed by the diagonal A C; and becauſe D C, A B, are parallel and equal, the right angled triangles A E B, D F C,
* *art.* 25. are * equal in all reſpects; ſo that B E and F C, or H A and A G, are equal: And ſince the forces expreſſed by H A, A G, are oppoſite and equal, they

they will deſtroy each other: And conſequently, AE, AF, expreſs the parts of the forces AB, AC, which act on the ſame body in the direction of the diagonal; and ſince AE is equal to DF, the ſum of theſe parts AF, AE, is equal to the diagonal. Which confirms, that a force expreſſed by and acting in the direction of the diagonal, is equivalent to the joint forces acting and expreſſed by the two ſides AB, AC.

COR. VI.

446. Laſtly, If a force AB acts upon a body, and it is required to find the part of that force which acts upon this body in any other direction DC; by drawing AD perpendicular to the direction DC, the abſolute force will be to the part relative to the direction DB, as AB is to DB, or as the radius to the coſine of the angle ABD, made by theſe directions. *Fig.* 3.

SCHOLIUM.

It may eaſily be proved from what has been ſaid, that relative motion obſerves the ſame laws as abſolute motion; for if we ſuppoſe that the figure ABCD moves with an uniform motion in the direction AB, and a force acts upon a body in the direction AC; it is plain that whilſt the line AC moves from A to E, or B, the body will by the impreſſed force move from E to F, or to D, in the line AC: So that a ſpectator ſtanding at A will ſee the body move in a right line AC; for when the line AC comes into the poſition BD, the ſpectator will be at B, and ſtill ſee the body move in the ſame direction, although the body moves in reality over the diagonal AD; and hence, whether our earth be in motion or at reſt, the motions of bodies will appear to us al- *Fig.* 1.

ways in the ſame manner, becauſe we move the ſame with the reſt of the objects placed on the ſurface.

Of MOTION, *uniformly accelerated or retarded.*

Notwithſtanding that gravity is different at different heights from the ſurface of the earth, and farther or nearer the equator, tho' equally diſtant from the ſurface; nevertheleſs, the difference is ſo little at a ſmall diſtance from the ſame point of the ſurface, that it produces no ſenſible error in computation; we ſhall therefore ſuppoſe with *Galileus* and others, that gravity acts uniformly and conſtantly the ſame in every part of time.

THEOREM VI.

447. *If a body begins to fall freely from reſt, it will acquire velocities proportional to the times, from the beginning of the fall.*

Suppoſe the time to be divided into very ſmall equal particles or inſtances; then as in each particle the force of gravity makes equal impreſſions on the body by ſuppoſition, whatever force therefore the body receives from the impreſſion of gravity in the firſt inſtance, it muſt receive as much in every other inſtance; and ſince the impreſſions remain, the velocity acquired in the firſt inſtant will be doubled in the ſecond, tripled in the third, and quadrupled in the fourth, and ſo on continually through every inſtant of the fall. Therefore, ſetting aſide all reſiſtance, the velocity of a falling body, freely from reſt, will conſtantly increaſe in the ſame proportion as the times of its deſcent from the beginning of the fall.

COR.

COR.

448. Hence, if the line AB expreſſes the time of the fall, BC perpendicular to AB, the velocity acquired during that time, by drawing AC, and from any point D, in AB, DE parallel to BC; then will DE expreſs the velocity acquired in the time AD. For becauſe the velocities acquired are as * the times elapſed from the beginning of the fall; and by reaſon of the ſimilar triangles ADE, ABC, we have * AB: AD:: BC: DE. It follows that DE expreſſes the velocity acquired in the time AD. *Fig.* 4. *art. 447. *art. 69.

THEOREM VII.

449. *The ſpaces moved over by a body falling freely from a ſtate of reſt by the force of gravity, are as the times and velocities conjointly.*

Let the line AB be divided into equal parts indefinitely ſmall, and from each of thoſe diviſions ſuppoſe lines, as DE, drawn parallel to BC: It is evident, from what has been ſaid, that thoſe lines expreſs the velocities acquired in their reſpective times; and as the velocities acquired in very ſmall particles of time may be conſidered as uniform, the ſpaces deſcribed in theſe ſmall particles of time * will therefore be as the times and velocities: But the ſum of the ſpaces deſcribed in all the ſmall portions of time is equal to the ſpace deſcribed from the beginning of the fall; and as the ſum of all the rectangles of the lines, as DE, and the ſmall parts of AD, are equal to the triangle BDE; it follows, that the ſpace deſcribed in the time AD is to the ſpace deſcribed in the time AB, as the triangle ADE is to the triangle *art. 435.

ABC; that is, as the rectangle of AD and DE
* *art.* 91. to * the rectangle of AB and BC.

COR. I.

450. Since the similar triangles ADE, ABC,
* *art.* 88. are to each other as * the squares of their homologous sides AD, AB, or DE, BC; it is evident that the spaces described by a falling body, from a state of rest, are as the squares of the times elapsed from the beginning of the fall, or as the squares of the velocities acquired during the fall.

COR. II.

451. If the rectangle ABCF be compleated, then the space described uniformly with the velo-
* *art.* 435. city AF or BC, in the time AB, will be * as the rectangle ABCF; and the space described by the accelerated motion of a falling body, in the same time AB, as the triangle ABC; and since
* *art.* 35. the rectangle BF is * double the triangle ABC, it follows, that the space described uniformly with a velocity acquired by a falling body from a state of rest, is double the space described in the same time by the falling body.

REMARK.

It has been found by experiments, that a body falling in the latitude of *London* from a state of rest, describes a space of 16.1 feet, *English* measure, in a second of time; this being supposed, by what has been said above, we may find the space a falling body will describe in any other time. For example, the space described in 4 seconds, will be found by saying, as the square 1 of 1 second is to the square 16 of 4 seconds, so is the

the ſpace 16.1 feet deſcribed in 1 ſecond to the ſpace 257.6 feet deſcribed in 4 ſeconds.

On the contrary, we find the time required by a falling body ſo as to deſcribe any ſpace 144.9, by ſaying as 16.1 feet is to 144.9 feet, ſo is the ſquare of one ſecond to the ſquare of the time required, which will be 9, whoſe ſquare root 3 will be the time required.

THEOREM VIII.

452. *If a body be thrown directly upward, the time of its riſe will be equal to the time* AB, *wherein a body falling freely from a ſtate of reſt acquires the ſame velocity* BC, *wherewith the body is thrown up.*

For ſince the action of gravity is conſtant and uniform in whatever time it generates any velocity in a falling body, in the ſame time, in a riſing body, by reaſon of its acting in a contrary direction, it muſt deſtroy that velocity: Therefore, if the time AB of a falling body be divided into equal particles or inſtances, whatever velocity is acquired in each part of time in the falling body, the ſame velocity is deſtroyed in the raiſing body in the ſame time: And conſequently, at the end of the time AB, the whole velocity BC of the raiſing body will be deſtroyed.

THEOREM IX.

453. *The height to which a body thrown directly upward riſes, is equal to the height from which a body falling freely from a ſtate of reſt, acquires a velocity equal to that wherewith a body is thrown upward.*

For ſince the times in which the velocity of a falling body is generated, and that of a raiſing body

body deſtroyed, are equal; and becauſe of the two equal velocities, one is generated and the other deſtroyed, by the conſtant uniform action of gravity: It is manifeſt, that whatever the ſpace through which the falling body moves, in order to acquire its velocity, the riſing body muſt aſcend through an equal ſpace in order to loſe its velocity; that is, it muſt riſe to the ſame height from which the other falls.

THEOREM X.

454. *All bodies falling from a ſtate of reſt, through equal ſpaces, acquire by the force of gravity equal velocities; that is, two bodies of different weights acquire the ſame velocity in the ſame time, ſetting aſide all external obſtacles and impediments.*

Suppoſe two bodies of the ſame kind, one of which being double in weight to the other, if they are let fall from the ſame height, at the ſame time, they will both together arrive at the ſame inſtant to the ground; for the greater body may be divided into two bodies, each equal to the leſſer, which being let fall together will deſcribe equal ſpaces in the ſame time; but whether this body be ſeparated into equal parts or not, it is evident that the force of gravity acts the ſame; therefore a body, double another body, falls with the ſame velocity.

If one body is to the other in any proportion, the greater may be always divided into a certain number of equal bodies which are equal to the leſſer; and if theſe parts are let fall together, they will paſs over equal ſpaces at the ſame time; but as the gravity of the parts is equal to the gravity of the whole, the forces in the parts ariſing from the

the action of gravity, must be equal to the force of gravity in the whole. Consequently, all bodies accelerate equally at the same time.

It is also confirmed by experiment, that all bodies, light or heavy, fall equally swift, in an exhausted receiver, as gold and a feather: So that if there is any difference in the open air, it is owing to the resistance of the medium in which they move.

Of the COLLISION *of hard non-elastic bodies.*

By hard non-elastic bodies we mean, those which, if they strike any obstacle, do not change their figure, and are void of all springs or elasticity.

THEOREM XI.

455. *Reaction is always equal to action; that is, the actions of two bodies upon one another are always equal and in contrary directions.*

For whatever change is made in the state of one body, whether at rest or in motion, by the action of another, the same change is produced in the state of the other by the reaction of the former; and the directions of those changes are contrary to each other.

THEOREM XII.

456. *The momentums or forces with which two unequal bodies* A *and* B *strike another body* C, *either in motion or at rest, with the same velocity, are as the weights of those bodies.*

If the body A be double the body B, it is plain that it may be divided into two bodies, each equal to B; and these two bodies having the same velocity

velocity as B, would each ſtrike the body C with the ſame force as the body B; but whether the body A be ſeparated into parts, or united in one, it will ſtrike the body C with the ſame momentum. If the body A be triple the body B, it will, by the ſame reaſon, ſtrike the body C with a force triple to that of the body B; and therefore, whatever the proportion is between the bodies A and B, they will, becauſe the effects are proportional to their adequate cauſe, ſtrike the body C with forces that are in the ſame proportion.

THEOREM XIII.

457. *If two equal bodies,* A *and* B, *ſtrike another body* C, *either in motion or at reſt, with unequal velocities, their momentums will be as the velocities.*

For if the velocity of the body A be double the velocity of the body B, it will ſtrike the body C with twice the force to that of the body B; if its velocity is triple, with a force triple; if quadruple, with a force quadruple; and ſo on, in the ſame proportion of the velocities. For ſince the effects are proportional to their adequate cauſes, the bodies being the ſame, the effects or forces muſt be proportional to the cauſes or velocities.

THEOREM XIV.

458. *If two unequal bodies,* A *and* B, *ſtrike another body* C, *either in motion or at reſt, with unequal velocities; their momentums will be as their weights and velocities conjointly.*

For if W, *w*, expreſs the weights of the bodies A and B; V, *v*, their velocities; and M, *m*, their momentums reſpectively; let there be another body D, whoſe weight is W, velocity *v*, and momentum

x;

x; then because the bodies A and D, having the same weight W, their momentums M, x, are * as the velocities V, v; that is, $M : x :: V : v$; and since the bodies B, D, have the same velocity v, their momentums m, x, are as * their weights w, W; *viz.* $x : m :: W : w$. If the terms of these proportions are multiplied together, we shall have $Mx : mx$, or * $M : m :: WV : wv$. Consequently, * art. 65. the momentums of two unequal bodies A, B, striking another body C, either in motion or at rest, with unequal velocities, are as their weights and velocities conjointly.

* art. 457.

* art. 456.

COR. I.

459. Hence, if the momentums M, m, are equal, we have $WV = wv$ by equality of ratios, or $W : w :: v : V$. Consequently, when the momentums are equal, the weights are reciprocally as their velocities; and, on the contrary, if the weights are reciprocally as the velocities, the momentums are equal.

COR. II.

460. It is also manifest, that the weights are as the momentums directly, and the velocities inversely: For if the antecedents in $M : m :: WV : wv$, are divided by V, and the consequents by v, we shall have * $\frac{M}{V} : \frac{m}{v} :: W : w$. And by the same way of reasoning the velocities are as the momentums directly, and the weights inversely.

* art. 54.

THEOREM XV.

461. *If a non-elastic hard body* A *strikes another body* B *of the same kind, either in motion or at rest; the sum or difference of their momentums will*

will remain the same after the stroke, as it was before, and they will move together in the direction of the body which had the greatest motion before the stroke.

*art. 455. For since reaction is always equal to * action, and in a contrary direction when the bodies move the same way, the one loses as much of its motion as it communicates to the other; and when they move contrary ways, the body which had the greatest motion, will lose as much of it as the other had before the stroke, and they will continue to move after the stroke in the direction of the body which had the greatest motion before the stroke. If the body B is at rest, the body A will lose as much of its motion as it communicates to B: Therefore they will continue to move with the sum of their momentums, if they went the same way, or with their difference if the contrary ways, and in the direction of the body which had the greatest motion before the stroke.

COR. I.

462. Hence, because the bodies A and B move together after the stroke, whether they moved before the same or contrary ways, or the body B was at rest; it is manifest, that they will both have the same velocity after the stroke.

COR. II.

463. Since the momentums are in the compound * ratios of the weights and velocities, and the bodies have the same velocity after the stroke; it is plain that the sum of the momentums before the stroke divided by the sum of the weights, will give their common velocity after the stroke, if they moved the same way; and the difference between

*art. 458.

between their momentums divided by the sum of their weights, if they moved contrary ways; but if B was at rest before the stroke, the momentum of A, divided by the sum of the weights, will give their velocity after the stroke.

If A and B express their weights, a, b, their respective velocities before the stroke, and v their common velocities after the stroke; then will

$v = \frac{a\mathrm{A} + b\mathrm{B}}{\mathrm{A} + \mathrm{B}}$, when they move the same way.

$v = \frac{a\mathrm{A} - b\mathrm{B}}{\mathrm{A} + \mathrm{B}}$, when they move contrary ways.

$v = \frac{a\mathrm{A}}{\mathrm{A} + \mathrm{B}}$, when B is at rest before the stroke.

Examp. Let the weights A and B be as 3 to 1, and their velocities a, b, as 6 to 2; then will 18, 2, be as their momentums, and 4 the sum of their weights: Therefore $v = \frac{18 + 2}{4} = 5$, in the first case; $v = \frac{18 - 2}{4} = 4$, in the second, and $v = \frac{18}{4} = 4.5$, in the third.

COR. III.

464. Since the body A loses as much of its motion by the stroke as B gains by it; and when two quantities of motions are equal, the weights are * reciprocally as the velocities; it follows, that if x be the velocity lost by A, and y that gained by B, we have $\mathrm{A} : \mathrm{B} :: y : x$, and by composition $\mathrm{A} + \mathrm{B} : \mathrm{B} :: y + x : x$; or $\mathrm{A} + \mathrm{B} : \mathrm{A} :: y + x : y$. Consequently, the sum of the bodies is to either, as the

* art. 459.

the ſum of the velocities gained and loſt is to the velocity of the other body.

COR. IV.

465. It is evident, that the body A ſtrikes the body B with the difference $a-b$ between their velocities, when they move the ſame way; or with their ſum $a+b$, when they move contrary ways; and only with its own when B is at reſt; and as it is this velocity which is divided between the bodies A and B after the ſtroke, $x+y$ is equal to $a-b$ in the firſt caſe, to $a+b$ in the ſecond, and to a in the third. Conſequently, $A+B:B::a\mp b:x$, and $A+B:A::a\mp b:y$, in both caſes, and b is nothing in the laſt.

The ſame velocities are alſo found by taking the differences between the common velocity v after the ſtroke, and the velocities a and b, which they had before the ſtroke.

Of the COLLISION *of* ELASTIC BODIES.

Bodies are ſaid to be Elaſtic, if they conſiſt of ſuch parts as yield and give way when preſſed, and reſtore themſelves upon the removal of the preſſure.

If the reſtoring force is equal to the force of compreſſion, then are theſe bodies ſaid to be Perfectly Elaſtic; but if they are unequal, the ratio of theſe forces is called the Elaſtic Ratio.

It is to be obſerved that all homogeneous bodies of the ſame kind, have their elaſtic ratio conſtant and invariable: For whatever the cauſe of elaſticity may be, it muſt be the ſame in all bodies whoſe conſtituent parts are the ſame.

THE-

THEOREM XV.

466. *If two elaſtic bodies* A, B, *move the ſame way, or the body* B *be at reſt, the ſum of their momentums after the ſtroke will be always equal to the ſum of their momentums before the ſtroke.*

For becauſe reaction is always equal to action, and in a contrary direction, if the bodies move the ſame way, whatever motion A loſes by ſtriking B, the ſame motion will be gained by B in the ſtroke, whence the ſum of their motion will remain the ſame in that inſtant; and whatever the force is, by which the parts reſtore themſelves to their former ſtate, it muſt act in the ſame manner on both; therefore, the body A loſes as much by the reſtoring force as B gains by it. Conſequently, the ſum of their momentums after the ſtroke, is equal to the ſum of their momentums before the ſtroke.

COR. I.

467. Since the quantity of motion loſt by the body A, by the force of compreſſion, is equal to the quantity gained by B; it is evident, that the body A is to the body B, as the velocity gained by B is to * the velocity loſt by A; and therefore, by compoſition, * $A+B:B::a-b:x$, and $A+B:A::a-b:y$. * art. 459. * art. 75.

COR. II.

468. It is evident, that the velocities loſt and gained conſiſt of two parts, the one ariſing from the force of compreſſion, and the other from that with which the parts reſtore themſelves, which parts are equal in perfect elaſtic bodies, and in a given ratio in non-perfect elaſtick bodies.

If a', b', expreſs the velocies of A and B after the ſtroke, and unity is to r, as the compreſſing force is to the reſtoring force; then becauſe $A+B : B :: a-b : \frac{B \times \overline{a-b}}{\overline{A+B}}$ = to the velocity of A, loſt by the compreſſing force, and $A+B : A :: a-b : \frac{A \times \overline{a-b}}{\overline{A+B}}$ to the velocity of B, gained by that force, their reſpective forces, loſt and gained by the reſtoring force, will be $\frac{r B \times \overline{a-b}}{\overline{A+B}}$, and $\frac{r A \times \overline{a-b}}{\overline{A+B}}$; adding the two forces loſt by A, we ſhall get $\frac{\overline{1+r} \times B \times \overline{a-b}}{\overline{A+B}}$, for its velocity loſt, and $\frac{\overline{1+r} \times A \times \overline{a-b}}{\overline{A+B}}$, for the total gain by B. Conſequently,

$$a' = -\frac{\overline{1+r} \times B \times \overline{a-b}.}{\overline{A+B}}$$

$$b' = b + \frac{\overline{1+r} \times B \times \overline{a-b}.}{\overline{A+B}}$$

If the bodies are perfectly elaſtic, then will $r = 1$; and therefore

$$a' = a - \frac{2B \times \overline{a-b}.}{\overline{A+B}}$$

$$b' = b + \frac{2A \times \overline{a-b}.}{\overline{A+B}}$$

And

And if the bodies are void of elasticity, we shall find the same expression for both velocities as before; therefore this theorem is general, and extends to all cases that can happen in the collision of elastic, or non elastic bodies: When the bodies move contrary ways, we need but change the sign of b, into $+$ when $-$, or into $-$ if $+$, in order to have the different expressions for that case.

COR. III.

469. If the perfect elastic bodies A and B are equal, and the body B was at rest before the stroke; by making $A = B$, and $b = o$, we shall have $a' = o$, and $b' = a$; which shews, that the striking body A remains at rest after the stroke, and the body B flies off with the velocity a of the former.

COR. IV.

470. If the perfect elastic bodies are equal, and move contrary ways, by making $A = B$, and changing the signs of b, we get $a' = -b$, and $b' = a$; which shews that both bodies reflect backwards and change their velocities.

COR. V.

471. If the body A strikes the body B at rest, with the velocity a, and the body B strikes another body C at rest, with the velocity b'; then if c' expresses the velocity acquired by C, we shall have $c = \frac{\overline{1+r}\,B\,b'}{\overline{B+C}}$: or because $b' = \frac{\overline{1+r}\,A\,a}{\overline{A+B}}$, there will be $c' = \frac{\overline{1+r^2} \times AB\,a}{\overline{A+B} \times \overline{B+C}}$.

We ſhall ſhew hereafter, when we treat of the *Maximums* and *Minimums*, that when $\div A : B : C$, the body C will acquire the greateſt velocity poſſible, by ſuppoſing the body A to be the leaſt; and from thence it follows, that when a body ſtrikes another at reſt, and this another, and ſo on to any number of bodies; and if theſe bodies are in an increaſing geometrical progreſſion, the laſt body will acquire the greateſt velocity poſſible.

SECT. II.

Of MECHANICS.

DEFINITIONS.

1. MECHANICS is the ſcience which conſiders the proportions between powers that act upon bodies, and the effects they produce when applied to machines. *Fig.* 18.

2. Two or more powers are ſaid to be in Equilibrio when their mutual actions, on the ſame object, deſtroy each other ſo as to remain at reſt.

3. Static is that part of mechanics which conſiders the proportions of powers in a ſtate of equilibrio.

4. The direction of a power is the line in which it acts upon the body, or compels it to move in.

5. All kinds of inſtruments which ſerve to move bodies, or to ſtop their motions, are called Engines, or Machines; there are ſeveral ſorts, ſimple and compound.

6. The

6. The ſimple are five, *viz* the Lever, the Pully, the Wheel, the Inclined Plane or Wedge, and the Screw.

The compound machines are without number, but are all made by the combination of the five ſimple ones.

7. The abſolute force of any power, is that which when applied either to move or to ſtop bodies in motion, ſuffers no diminution from the direction in which it acts.

8. The relative force of a power is that part of the abſolute force, which ſerves to produce the deſired effect, while the other part is deſtroyed by ſome oppoſition.

AXIOM.

Gravity acts conſtantly the ſame, within a ſmall diſtance of the ſurface of the earth, and in directions parallel to each other.

Though this is not true, according to a geometrical exactneſs, yet it anſwers ſo nearly, that by the niceſt computations we can make, there is no ſenſible difference to be found; for which reaſon *Galileus*, and other authors, admit this axiom as an undeniable truth.

THEOREM I.

472. *If two powers* P, Q, *ſuſtain a weight* W, *by means of ſtrings, ſo as to remain at reſt, they will be to each other as the ſides of the parallelogram made of their directions, that of the weight being in the diagonal.* Plate XVI. Fig. 5.

For let A D B C be the parallelogram, deſcribed on the directions C B, C A, of the powers, and C D that of the weight; then becauſe the forces expreſſed by and in the directions of the two ſides

* *art.* 442. CA. CB, are * equivalent to the force expreſſed by and in the direction of the diagonal; and the weight W acts downwards from D to C, while the powers P, Q, act from C to A and B; the action of the weight being equivalent to the joint actions of the powers, and in a contrary direction, they muſt deſtroy each other, and therefore remain in a ſtate of reſt. Conſequently, theſe powers P, Q, are to each other as the ſides CA, CB, of the parallelogram.

COR.

473. Hence, as the angle ACB made by the directions of the powers P, Q, opens, ſo theſe powers muſt increaſe, in order to ſuſtain the weight W, which we ſuppoſe to remain the ſame; and when the directions CA, CB, become one continued right line, it is evident that the powers P and Q muſt be infinite to ſupport the leaſt weight W poſſible; which ſhews that no finite power is able to ſtretch a rope of any length, ſo as to make it ſtrait.

THEOREM II.

Fig. 6 474. *If three powers* P, Q, R, *fixed to an inflexible line* AB, *by means of ſtrings, are in equilibrio; the directions of the two powers* Q, R, *on the ſame ſide of this line, will, when produced, meet the direction of the other power* P *in the ſame point* D, *if they are not parallel.*

From the point C, where the power P is fixed to the inflexible line, draw CE and CF parallel to the directions AD, BD, of the powers Q, R; then as it is the ſame thing whether theſe powers are fixed to the line AB, or to any other points in their directions, it is evident, that theſe powers act in the

the same manner as if they were fastened together by means of strings, AD, BD, and CD; and therefore they will be to each other * as the sides CE, CF, of the parallelogram made of their directions, since they are in equilibrio by supposition. But this cannot happen unless the directions AD, BD, of the powers Q and R meet the direction CD of the opposite power P, in the same point D. Consequently, their directions must meet in the same point when produced. * art. 472.

COR. I.

475. Since * R : Q :: ED : FD; or because CF = ED, and CE = FD, this proportion will be R : Q :: CF : CE. Consequently, two powers Q and R are in equilibrio with a third P, when they are reciprocally as the lines CE, CF, drawn from the point fix C parallel to their directions, and terminated by the same. * art. 472.

COR II.

476. If from the point C the lines CL, CK, are drawn perpendicular to the directions BD, AD; then because CF, LE, and CE, KF, are parallel by supposition, the angles CFK, CEL, formed by these parallel lines, are equal; and therefore the right angled triangles CKF, CLE, are * similar; whence CF : CE :: CK : CL: And since * R : Q :: CF : CE, we have R : Q :: CK : CL, by * equality of ratios. Consequently, the powers Q, R, are also reciprocally as the perpendiculars drawn from the point C to their directions, when they are in equilibrio. * art. 67. * art. 475. * art. 53.

COR. III.

477. It is manifest, that if any two directions of the three powers which are in equilibrio are

parallel to each other, the third must likewise be parallel to these two. For should AD and BD be parallel, and CD meet any of the two former, it is evident that these powers * could not be in equilibrio, which is contrary to supposition.

* *art.* 474.

COR. VI.

478. Hence, if the directions of three powers which are in equilibrio, are parallel, the two powers Q, R, on the same side of the inflexible line AB, will be reciprocally as the parts of that line, terminated by the point C, and their directions.

Fig. 7. For if the directions are perpendicular to the line AB, the perpendiculars CK, CL, in fig. 6, will coincide with the parts CA, CB; and there-
* *art.* 476. fore the proportion * R:Q::CK:CL becomes R:Q::CA:CB.

Fig. 8. And when the directions are oblique to the line AB, these perpendiculars CK, CL, become one continued right line; and since the right angled
* *art.* 69. similar triangles CKA, CLB, give * CK:CL::
* *art.* 476. CA:CB; and because * R:Q::CK:CL, we have R:Q::CA:CB, by equality of * ratios.

N. B. As this proportion is of the greatest use in mechanics, we shall give another demonstration of it, independent of what has been said before.

THEOREM III.

Fig. 9. 479. *If two weights* Q, R, *are suspended by an inflexible line* AB, *which is supported at the point fix* C, *they will be in equilibrio, if they are reciprocally as the distances* CA, CB, *of their directions from the point fix* C.

Produce the line AB both ways, so as AE be equal to BC, and BF to AC; take BD equal to BF

BF or to AC; then will AD be equal to AE or BC, and DE will be double the line BC, as likewiſe DF double the line AC: Now if the weights Q, R, are reciprocally as AC, BC, we have Q : R :: (2 BC : 2 AC ::) : DE : DF. Hence, if DE expreſſes the weight Q, DF will expreſs the weight R; and becauſe the lines DE, DF, are biſected in A and B, if they were ſuſpended in thoſe points, there is no reaſon for their inclining more one way than another; and ſince EA is equal to CB, and AC to BF, by conſtruction, the line EF is biſected in C; and therefore, if this be ſuſpended by the point C, it will alſo remain in equilibrio. Conſequently, if inſtead of DE the weight Q be ſuſpended to the point A which biſects it; and inſtead of DF the weight R be ſuſpended at the point B; then becauſe Q : R :: DE : DF, theſe weights will have the ſame power to move the line AB, as the lines DE and DF, which are proportional to them; but we have proved that the line EF will be in equilibrio, if ſuſpended by the point C, which biſects it. Conſequently, the powers Q and R, which are reciprocally as the diſtances CA, CB, of their directions from the point C, will alſo be in equilibrio.

COR. I.

480. Since we have always R : Q :: CA : CB, when the powers or weights Q and R are in equilibrio, and * CA × Q = CB × R; it follows, that the momentums of powers ſuſpended by an inflexible line are in the compound ratio of their forces and the diſtances of their directions from the point of ſuſpenſion C. *Fig.* 7. * *art.* 72.

COR.

COR. II.

Fig. 8. 481. It is alſo manifeſt, that if the inflexible line AB was moved about the point fix C; the vertical aſcent of one power will be to the deſcent of the other, reciprocally as theſe powers: For AK expreſſes the aſcent of the power Q, and BL the deſcent of the power R; and becauſe * *art.*476. * R : Q :: CK : CL, when theſe powers are in equilibrio, we have by the ſimilarity of triangles CK : CL :: AK : BL. Conſequently, R : Q :: AK : BL.

It is by this principle that *Deſcartes* has demonſtrated his mechanics: But tho' it has been proved before, that if the momentums of ſtriking bodies are equal, and in contrary directions, their motions will be deſtroyed; yet it does not appear clearly from thence, that bodies or powers fixed together, which are reciprocally as the diſtances of their directions from the point fix, ſhould be in equilibrio; for which reaſon we chuſe to demonſtrate the principles of mechanics in the uſual manner, and to deduce this by way of corollaries.

COR. III.

482. Since R : Q :: AC : CB, we have by compoſition R + Q : R :: AB : AC; that is, the ſum of the weights is to any one of them, as the length of the line AB is to the diſtance of the direction of the other from the point of ſuſpenſion C. Conſequently, the weights and the length of the line AB being given, the point of ſuſpenſion may be found.

Examp. Let the weight Q be 8 pounds, and the weight R, 6; and let the line AB be 12 feet; then

then the proportion above gives $14 : 6 :: 12 : AC = 5\frac{1}{7}$. If one of the weights Q, be 20 pounds, and the distances CA, CB, are 4 and 5 feet respectively; the other weight R will be found by this proportion, $BC(5) : CA(4) :: Q(20) : R = 16$ pounds.

REMARK.

It is upon this principle that the common scales for weighing all sorts of commodities are made; for when the weights Q and R are equal, the distances CA, CB, are also equal; but as it often happens that one arm is made longer than the other, and the weight of the other side is somewhat greater, in order to make the scale hang even, by which the buyer is often imposed upon; this may be easily known; for when the goods are weighed, if you put them on the other side where the weight was, and the weight in their place, then if the scale is not true you will easily discover the cheat.

The *Roman* balance, or steelyard, is likewise made from the foregoing principles; its construction is as follows: Take an iron rod FB, somewhat tapered at one end F, and suspend it at a point C, pretty near the heavier end B; suspend any weight R to the end B, and any other P at A, so as to slide along the rod; move this weight backwards and forwards till it balances the weight R, and mark the rod at the point of supposition with the number 1; then suspend another weight at R, double the former, and slide the weight P till it balances the other, and mark the rod with the number 2; continue thus to suspend successively weights at R, such as are to one another as the natural numbers 1, 2, 3, 4, *&c.* and mark every *Fig.* 10.

every point of ſuſpenſion of the weight P, with theſe numbers 1, 2, 3, 4, *&c.* and you will have the ſteel-yard required.

If the firſt weight ſuſpended at R be 1, 2, 3, 4, 5 pounds, the weight P ſuſpended at the point marked 1, will weigh the ſame number of pounds; and when ſuſpended at number 2, 3, 4, it will balance a weight, 2, 3, 4 times the former; by which one ſingle weight P will ſerve to weigh from a ſmall to a conſiderable weight.

THEOREM IV.

Fig. 11. 483. *If ſeveral weights* P, Q, R, S, *are fixed to an inflexible line, in ſuch a manner that the ſum of the momentums of all the weights on one ſide of the point fixt, is equal to the ſum of the momentums of all thoſe on the other; then they will be in equilibrio.*

* *art.* 480. For we have proved * that the momentums of weights or powers, were in the compound ratios of their weights or forces, and the diſtances of their directions from the point of ſuſpenſion: If therefore the ſum of the momentums of the weights P and Q, on one ſide of the point C, is equal to the ſum of the momentums of the weights R, S, on the other, theſe weights will remain in equilibrio, ſince there is no reaſon for their moving one way or the other.

COR.

484. Hence, if there be a rod AB of iron, or of any other metal, ſuſpended by the point C, and a weight P fixed at one end, we may find the weight S, which being ſuſpended at the other end B, ſo as theſe weights, together with the rod, ſhall be in equilibrio: For if the points D and

E

E are the centers of gravity of the parts AC, CB, of the rod; and the weights Q, R, expreſs the weights of theſe parts; then by taking the momentums of the weights P, Q, and ſubtracting from their ſum the momentum of the weight R, the difference divided by the diſtance CB will give the weight required.

THEOREM V.

485. *If the point of ſuſpenſion* C *be either above or below the line* AB, *and the weights* P *and* Q *are in equilibrio, when the line* AB *is in an horizontal poſition, they will not be ſo in any other.* Fig. 12, 13.

Let CD be drawn perpendicular to AB, then will P : Q : : DB : AD, when theſe weights are in equilibrio; but if the line AB is moved into any other poſition, ſuch as *a b*, and CD is repreſented by the line C*d*, draw *d*H perpendicular to AB. Now if we ſuppoſe the weights to be reunited into the point D, that is, into their common center of gravity, when AB is in an horizontal poſition, they will remain at reſt; but when that line inclines, the direction of theſe weights will be in *d*H, which being oblique to C*d*, will make the weight Q deſcend, when the point C is below the line AB, as low as poſſibly it can, and never returns again into the former poſition; and when the point C is above the weight, Q will deſcend and aſcend alternately till ſuch time that the line AB remains in an horizontal poſition.

N. B. In common ſcales the point of ſuſpenſion C is generally a little above the beam; becauſe when the weights are equal, they will ſoon come to an equilibrio; but as a ſmall difference between the weights is imperceptible, that point ſhould be placed exactly in the beam, in ſcales uſed for weighing ſilver, gold, and other valuable things.

And

And in draw-bridges, the point C is generally placed below the axis, about which it turns, on account of their being raiſed with a leſſer force; but when it comes to a certain height, its motion accelerates to ſuch a degree, as to ſhake the whole building by its ſhock; for which reaſon this practice ſhould be avoided.

THEOREM VI.

Fig 14. 486. *If two weights* P, Q, *are fixed to a crooked inflexible line* ADF; *they will ſtill be reciprocally as the diſtances of their directions from the point of ſuſpenſion, if they are in equilibrio.*

Let the ſtrait part AD be produced ſo as to meet the direction of the weight Q in B; then as it is the ſame thing whether the weight Q be fixed to the point F or B, it is evident that the weight P is to the weight Q as * BC is to CA, if they are in equilibrio.

**art.* 479.

Of the LEAVER.

The Leaver is an inflexible rod or beam, ſupported in a point by a prop, which ſerves to raiſe weights: When the power is applied to one end, the weight fixed to the other, and the prop between them, it is called a leaver of the firſt kind, but when the weight is between the power and the prop, it is then called a leaver of the ſecond kind: Some authors diſtinguiſh five different ſorts, without reaſon; for as the power is always applied to one end, either the prop or the weight muſt be at the other; and therefore there are but two ſorts in nature.

The leavers are ſuppoſed to be void of gravity, though there is no ſuch thing in nature; becauſe

if

if their weight was to be considered, it would render the demonstrations more tedious, without any advantage in practice.

THEOREM VII.

487. *In both kind of leavers the power and the weight are reciprocally as the distances of their directions from the prop.* Fig. 15.

I. Let AB be a leaver of the first kind, and suppose the direction of the power P to be parallel to that of the weight; then will P : W :: CA : CB, by *art.* 480. But if the direction of the power be oblique to that of the weight; it is plain, that the power will then be to the weight as the distance CA between the direction of the weight and the point C, is to * the distance between the direction of the power P and the point C. *art. 476.

II. Let BC be a lever of the second kind; then if we consider the prop C, as another power R, which with the power P supports the weight; we shall have * P : : R : : AC : BA; and by composition P : P + R : : AC : BC. Now because the powers P and R support together the weight W, the sum of their efforts must necessarily be equal to that weight; if therefore we substitute the weight W, instead of the sum P + R of the powers, in the last proportion, we shall have P : W : : AC : BC. Fig. 16. *art. 479.

If the direction of the power P be oblique to that of the weight, the power P will still be to the weight W, reciprocally as the distance between the direction of the weight and the point C, is to the distance between the direction of the power and the same point C; since the momentum of any

*art. 480 any power * is in the compound ratio of its force, and the distance of its direction from the point fix.

COR.

488. Since in both leavers the power is always to the weight reciprocally as the distance between the direction of the weight, and the prop is to the distance between the direction of the power and the prop; a power applied to either leaver will have the same Advantage; if the distance of its directions from the prop is the same.

N. B. Upon this principle is founded the great use of handspikes, and other simple machines of the same nature; though the advantage seems to be the same, whether a power be applied either to the first or second kind; nevertheless, when the prop is between the power and weight, the power will be able to raise or support a greater weight than if the prop were at the end; because the weight of the leaver conspires with the action of the power in that case, whereas it acts in a contrary direction in the other.

Fig. 17. There is often used a long leaver AB, with a prop CE fixed to it by means of an iron pin C, about which it turns; when this leaver is used in artillery, or upon any other occasion to raise great weights, it is from 12 to 15 feet long, 3 or 4 inches diameter; but when it is only for raising small weights, the leaver is not above 5 feet long.

Plate XVII. *Fig.* 18. There is a machine used in *Germany* called Hebstock, made of one single piece of wood AB, with a slit or opening in the middle, and has holes across placed diagonally, which serves to raise great weights in the following manner. A leaver GH, with a wooden handle of about two feet long, has a hook with two or three links fixed

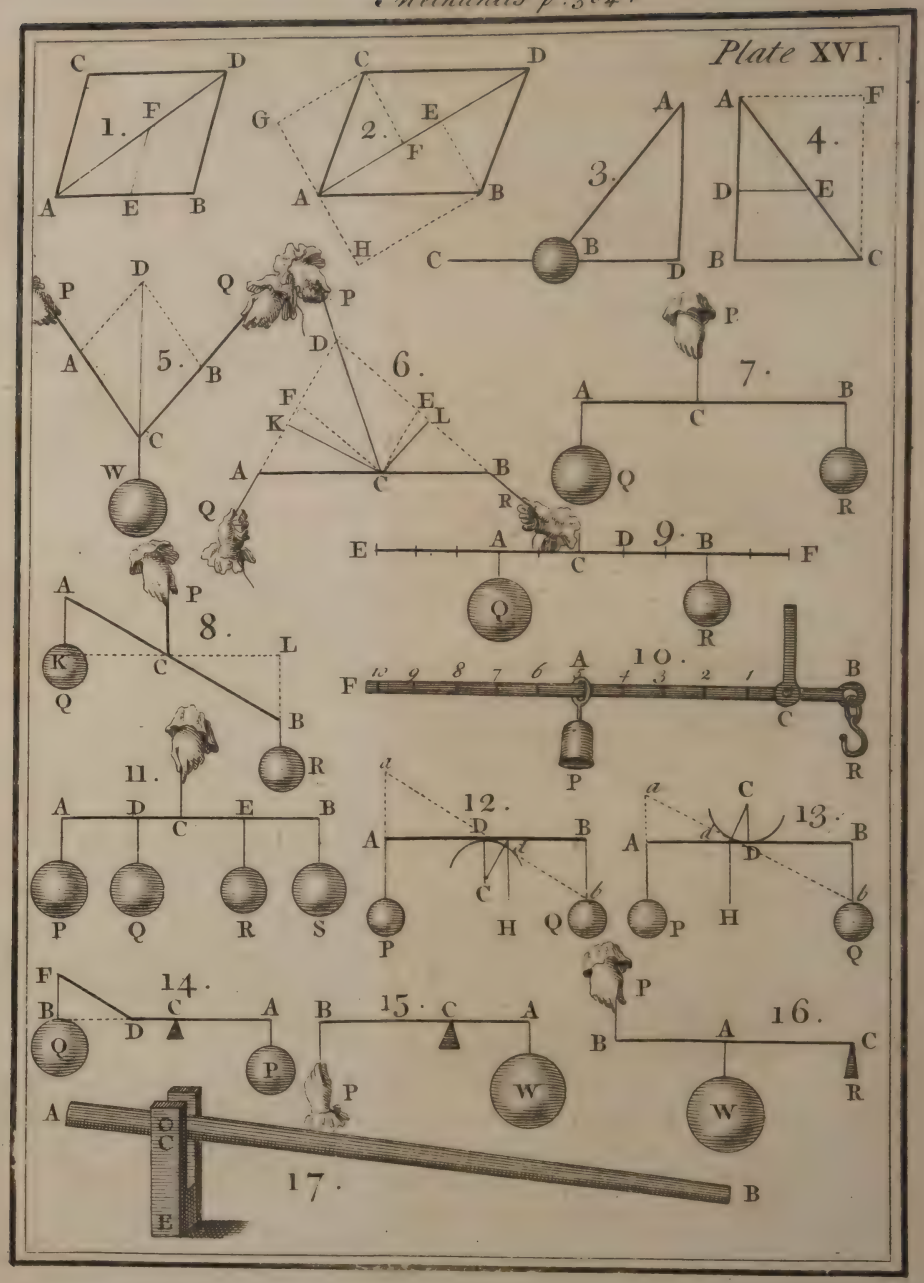
Plate XVI.
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fixed to the end, which is placed in the opening, and fastened on the opposite side to the burden with the hook; this done, there are two iron pins F, one of which is put into a hole under the leaver, as high as possible; then the end is raised so as the other pin may be put into the next hole, above the former; after this the end is pressed downwards, and the first pin is put higher: Thus, by raising and depressing alternately the end D of the leaver, the burden may be raised to any height required.

This machine is very simple, and costs but little; the country people make them themselves, and is of excellent use in loading of large timbers on waggons, which a single man and a boy may do; it may serve upon all occasions where a hand jack is used; and as it is more simple and easier repaired, it is better, and preferable to it.

Of the PULLEY.

The Pulley is a small wheel made of wood, or brass, and turns about an iron pin called the axis; which, when the center is fixed to an immoveable object, is called immoveable, and moveable when the weight is fixed to the center.

THEOREM VIII.

489. *If a power* P *sustains a weight* W, *by means of a rope passing over an immoveable pulley* AB, *in any direction, provided it be in the plane of the pulley; the power will sustain the whole weight.* Fig. 19.

For it is the same thing, whether the power and the weight are fixed to the extremities of the diameter AB, or the rope passes over the pulley;

ſince the power will, in both caſes, have the ſame advantage: Therefore this pulley may be conſidered as a leaver of the firſt kind, where the prop is the point C, or axis of the pulley. And ſince the radii, drawn perpendicular to the directions, expreſs their diſtances from the point C, and are equal by the property of the circle; it is evident, that the power P is * equal to or ſuſtains the whole weight W.

* *art.* 487.

COR.

490. Hence, if the power P raiſes the weight W by means of this pulley, the ſpace through which the weight aſcends, is equal to the ſpace through which the power deſcends; ſince one end of the rope is as much lengthened as the other is ſhortened.

Hence it appears, that this pulley is of no advantage in reſpect to the increaſe of power; yet it is nevertheleſs very uſeful in many occaſions: For it often happens that the direction of a power is not convenient, and by means of this pulley it may be changed into any other. Horſes can draw only in a horizontal direction, which cannot always be had but by means of this pulley; and if a man was to carry a burden to a great height, it will be more convenient to do it by means of this pulley than to carry it upon his back.

It may be obſerved, that the diameter of a pulley may be great or ſmall, without changing the equality between the power and weight; but the greater it is in reſpect to that of the axis, ſo much leſs will the friction be.

THEOREM IX.

Fig. 20. 491. *If a power* P *ſupports a weight* W *by means*

of a moveable pulley AB, *in a parallel direction, it will ſuſtain but half the weight.*

If the rope, fixed to an immoveable Object D, paſſes under the moveable pulley AB, and over the fixed one EF; then becauſe it is the ſame thing, whether the point or extremity B of the diameter AB is fixt, and the power applied to the other A in a direction parallel to that of the weight, or that the rope paſſes under the pulley; it is evident that this pulley may be conſidered as a leaver of the ſecond kind, where the prop is at one end and the power at the other; and therefore * P : W :: CB : AB, in caſe of an equilibrio: And becauſe the radius CB is half the diameter AB by the nature of the circle, the power P will ſuſtain but half of the weight W by equality of ratios.

* *art.* 487.

It may be obſerved, that by paſſing the rope over the firſt pulley EF, it does not change the proportion between the power and the weight; ſince that pulley does not increaſe the power, it only changes the direction from upwards to downwards, which is more convenient, and often neceſſary.

COR. I.

492. It is evident that if the power P moves the weight W, by means of this pulley; the ſpace through which the weight aſcends is but half that through which the power deſcends: For if the weight W is raiſed one inch, the rope BD is ſhortened by an inch, as well as the part AF; and therefore the part EP is lengthened by two: And, in general, whatever ſpace the weight W paſſes over, the parts BD and AF are ſhortened by as much, and the part EP is conſequently lengthened by twice as much.

COR. II.

493. If the direction of the power inſtead of being parallel was oblique, as AG to that of the weight, by drawing BH perpendicular to AG; it is plain that the power P is to the weight W, as * CB is to the perpendicular BH, to the direction of the power; ſince BH expreſſes the diſtance of the direction of the power, from the point B, which may here be conſidered as fixt.

*art. 476.

THEOREM X.

494. *If a power* P *ſuſtains a weight* W *by means of a windleſs containing ſeveral pulleys, one half of them being fix and the other moveable, ſo as all the parts of the rope are parallel to each other; the power* P *will be to the weight* W, *as unity to the number of all the pulleys.*

Fig. 21.

Becauſe the moveable pulleys ſuſtain the weight all together, they will ſupport equal parts; ſo that if there are three each will ſupport one third of the weight; if there are four, one fourth; and ſo on in the ſame proportion; and ſince the power ſuſtains but one half of the weight by means of one moveable pulley, it will ſuſtain one fourth, by means of two; one ſixth by means of three; and if there are four, each of them will ſuſtain one fourth; and the power one half of this, that is, one eighth. Conſequently, as the number of immoveable pulleys is equal to that of the moveable ones, the power is to the weight as unity to the number of all the pulleys.

COR.

495. Hence, if a power P raiſes a weight W, by means of a windleſs compoſed of any number of pulleys; the aſcent of the weight will be to the

the descent of the power, as unity to the number of all the pulleys. For suppose the weight to raise one foot, it is plain that each end of the rope of the moveable pulleys must be shortened by a foot; and as there are as many pulleys in all, as there are ends, the power P must descend as many feet as there are pulleys; and through whatever space the weight ascends, each end must be shortened by as much, and the power P descends thro' a space, twice as much as there are moveable pulleys Consequently, the space moved over by the weight is to the space moved over by the power, as unity to the number of all the pulleys.

It has been found by experiments that a man may raise 150 pounds; if this force be applied to a windless of three moveable pulleys, he may raise 6×150 or 900 pounds. By this one may easily conceive to what degree a force may be increased by means of these pulleys; but it must be considered that what is gained in power is lost in time; which, however, is convenient on many occasions, when power is wanting and not time.

Of the WHEEL.

THEOREM XI.

496. *If a power* P *be applied to a wheel in a direction of a tangent to its circumference, and sustains a weight* W, *fixed to its axis* AL; *the power will be to the weight reciprocally as the radius* CA *of the axis is to the radius* CB *of the wheel.* Fig. 22

It is evident, that whether the power be applied to the extremity B of the line AB, and the weight to the other A, or to the ropes which pass over

the wheel and round the axis, the power will have the ſame advantage; and as the line AB may be conſidered as a leaver of the firſt kind, where the prop C is between the power and the weight; the power P will be to the weight W, as the radius CA of the axis is to the radius CB of the wheel.

COR. I.

497. Hence, if the power P raiſes the weight W by means of the wheel, the aſcent of the weight will be to the deſcent of the power, as the radius CA of the axis is to the radius CB of the wheel: For ſince the wheel and axis move together, the ſpace moved over by the weight in one turn of the wheel, will be to the ſpace moved over by the power in the ſame time, as the circumference of the axis is to the circumference of the wheel; but becauſe the circumferences of circles are as their radii, the ſpace moved over by the weight is to that moved over by the power, as the radius CA of the axis is to the radius CB of the wheel.

COR. II.

498. If the direction of the power P is not in a tangent to the circumference, but in any other as DE; then by drawing CE perpendicular to DE, we ſhall have *P : W :: CA : CE. Therefore the power P acts to the greateſt advantage when its direction is a tangent to the circumference of the wheel.

*art. 476

The wheel is very uſeful on many occaſions; ſuch as in drawing water out of wells, either by hand or with horſes; ſtones out of pits; earth out of mines: In ſome caſes the circumference is omitted, and the power is applied to the extremities

mities of the radii; the machine is then called a Cabſtan.

Of the inclined PLANE and WEDGE.

THEOREM XII.

499. *If a power* P *ſuſtains a weight* W *by means of an inclined plane* AB, *in a direction* DP, *parallel to the plane; the power will be to the weight as the height* BC *of the plane is to its length* AB. Fig. 23.

Through the center of gravity D of the weight, draw the line DE perpendicular to the baſe AC, and DF, to the plane; from any point in DE draw EF parallel to AB, and FG parallel to ED; then if the ſide DE expreſſes the weight W, the diagonal DF of the parallelogram EG will expreſs the part ſuſtained by the plane, and EF that which is ſupported by the power P; for whilſt gravity acts in the direction of the ſide DE, the power in DG, the plane reſiſts in the direction FD of the diagonal. Now becauſe DE is parallel to BC, and EF to AB, the angle E will be equal to the angle B; and ſince DF is perpendicular to AB, it will alſo be perpendicular to its parallel EF: Therefore the triangle DEF is right angled at F and ſimilar to * the triangle ABC; and ſo EF:ED::BC:AB; but the weight W has been proved to be to the part ſuſtained by the power P, as DE is to EF. Conſequently P:W::BC:AB, by equality of ratios.

* *art.* 69.

COR.

500. Hence, if a power P raiſes a weight W, by means of an inclined plane, the vertical aſcent

of the weight is to the ſpace deſcribed by the power as the height of the plane BC is to its length. For it is evident, that whilſt the power moves the weight from A to B, it deſcribes the ſame ſpace AB, and the weight is raiſed from C to B.

THEOREM XIII.

Fig. 24. 501. *If a power* P *ſuſtains a weight* W *by means of an inclined plane* ABC, *in a direction* GP *parallel to the baſe* AC; *the power will be to the weight as the height* BC *of the plane is to its baſe* AC.

By compleating the parallelogram EG in the ſame manner as before, only ſo as EF be parallel to the baſe AC; then if DE expreſſes the weight, EF will expreſs the part ſuſtained by the power P, and DF that which is ſuſtained by the plane: And becauſe DE is parallel to BC, and EF to AC, the angle E will be a right one; and ſince DF is perpendicular to AB, it will make the angle EDF, with DE, parallel to BC, equal to the angle A; hence the triangles ABC and DEF will be the ſimilar: Therefore * EF:ED::BC:AC; but we have proved that the power is to the weight as EF is to ED. Conſequently P:W::BC:AC.

* *art.* 69.

COR.

502. Hence, if the power P raiſes the weight W, by means of an inclined plane, in a direction parallel to the baſe AC; the ſpace moved over by the power is to the vertical aſcent of the weight as the length of the baſe AC is to the height BC of the plane; for whilſt the power P moves the weight W from A to B, it moves itſelf through the ſpace AC, and the weight aſcends from C to B.

N. B.

N. B. If we ſuppoſe an equal plane was joined to this, ſo as to have the ſame baſe A C, we ſhall have the wedge; and the force in the direction C A, to drive it into a piece of wood, will be to its reſiſtance as twice E F is to twice D F, or as the height B C is to the length A B.

The vouſſoirs of arches may be conſidered as ſo many wedges which endeavour to deſcend by their gravity; and therefore the proportion between the reſiſtance and the weight of any one, may be determined when its poſition is given.

Of the S C R E W.

Although the ſcrew is well known, yet it will be neceſſary for the better underſtanding its application, to ſhew in what manner the thread of it may be formed.

Suppoſe the height A B of a cylinder A B C D *Fig.* 25. to be divided into any number of equal parts, B *a*, *a b*, *b c*, *c d*, and lines *a e*, *b f*, *c g*, *d h*, drawn perpendicular to the ſide A B, each equal to the circumference of the baſe; by joining B *e*, *a f*, *b g*, *c h*, there will be as many right angled triangles B *a e*, *a b f*, *b c g*, *c d h*, as the height of the cylinder has been divided into equal parts.

Now if we imagine theſe triangles to be rolled upon the cylinder, it is evident that the point *e* will coincide with the point *a*; *f* with *b*; *g* with *c*; *h* with *d*; and therefore the hypothenuſes B *e*, *a f*, *b g*, *c h*, will form on the ſurface of the cylinder one continued ſpiral line, which repreſents the thread of the ſcrew.

THEOREM XIV.

503. *If a power applied to a ſcrew ſuſtains*

a

a weight in a direction perpendicular to the axis and quite close to the circumference; the power will be to the weight, as the interval between any two contiguous threads is to the circumference of the screw.

Let the power P be applied to the point B in the direction BC; and suppose the screw to be unfolded so as to form the inclined planes B*e*, *af*, *bg*, *ch*; then it is plain that the power is to the weight as the * height *a*B of the plane is to its base *ae*; but the base *af* is equal to the circumference of the screw by the generation of it, and the height *a*B of the plane equal to the distance between any two contiguous threads. Therefore the power is to the weight as the interval between any two contiguous threads is to the circumference of the screw.

*art. 501.

COR.

504. It is evident, that if a power raises a weight by means of a screw, the space thro' which the weight ascends is to the space described by the power as the interval *a*B between any two contiguous threads is to the circumference *ae* of the screw.

Fig. 26, As the power is never applied close to the screw when it is made use of to raise weights, a leaver GH is passed through a hole made in the nut GI, and by turning it round it presses against a strong board, by which the screw presses downwards or upwards, according as the nut is below or above the board: It is plain, that in this case the power is to the weight as the interval between any two contiguous threads is to the circumference described by the point H, of the leaver, where the power is applied.

If

If the leaver GH be four feet, the circumference described by the point H will be 25 feet nearly, or 300 inches; and if the interval between two contiguous threads be an inch, the power will be to the weight as unity is to 300: Now if a man can carry 130 pounds, he may, with the same force applied to a screw, raise 130 x 300, or 39000 pounds nearly; for this is the weight which would be in equilibrio with that force, setting aside all friction and obstacles: Which shews to what prodigious degrees a power may be increased when applied to a screw.

THEOREM XV.

505. *If a power* P *sustains a weight* W, *by means of an endless screw* AB, *with a handle* ACD *to it*; *and this screw turns a toothed wheel* E, *and the pinion* L *of it, another wheel, and this continued to any number*; *and the weight* W *be fastened to the barrel* N *of the last wheel*; *the power will be to the weight as the product of the distance between any two contiguous threads, multiplied by the radii of all the pinions and the barrel, is to the product of the circumference described by the handle* CD, *multiplied by the radii of all the wheels.* Plate XVIII. Fig. 27.

If a power N were applied to the circumference of the lower wheel G, in the direction of a tangent; this power N would be to the weight W, as the radius of the barrel N is to the radius of the wheel; and if a power M were applied to the circumference of the second wheel F, it is evident that the power N may be considered as to be applied to the pinion M; and so the power M will be to the power N, as the radius of the pinion M s to the radius of the wheel F: In the same manner,

ner, if a power L were applied to the third wheel E, that power will be to the power M as the radius of the pinion L is to the radius of the wheel E.

Laſtly, the power L may be conſidered as applied to the endleſs ſcrew; and therefore the power P will be to the power L as the interval between any two contiguous threads, is to the circumference deſcribed by the handle.

Now if the radii of the barrel N, and the pinions M, L, be *n*, *m*, *l*, reſpectively, thoſe of the wheels G, F, E; *g*, *f*, *e*; the diſtance between any two contiguous threads of the ſcrew *d*; and the circumference deſcribed by the handle *c*; we ſhall have

N : W :: *n* : *g*.
M : N :: *m* : *f*.
L : M :: *l* : *e*.
P : L :: *d* : *c*.

P : W :: *nmld* : *gfec*.

COR.

506. If the power P raiſes the weight W, by means of theſe wheels and the ſcrew, the ſpace deſcribed by the power P will be to the ſpace deſcribed by the weight W, as the circumference deſcribed by the handle multiplied by the number of revolutions it makes, whilſt the barrel N turns once to the circumference of the barrel; or becauſe the circumferences are as their radii, as the radius of the handle multiplied by the number of revolutions to the radius of the barrel.

If the number of teeth in the wheels and the pinions are proportional to their radii or circumferences;

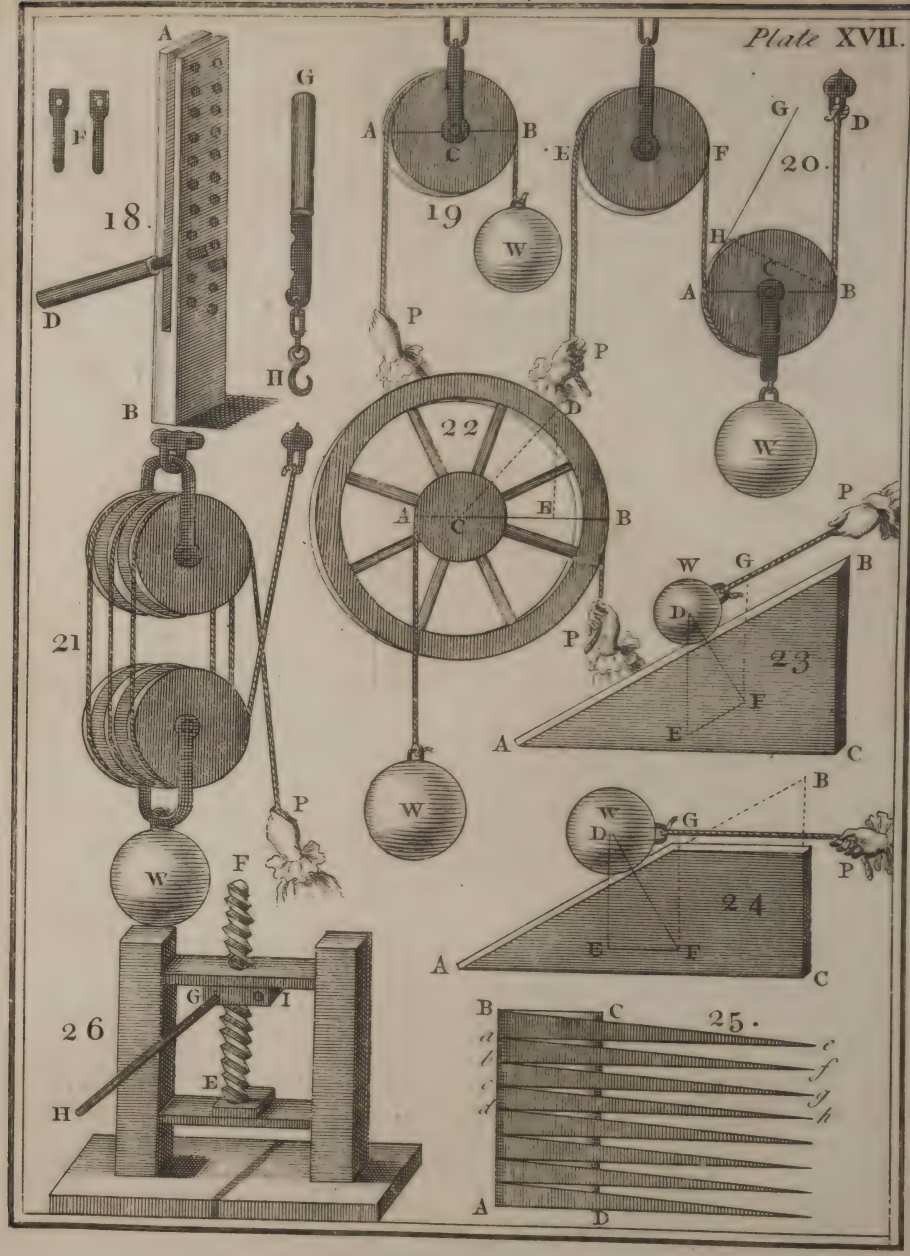
Plate XVII.
18.
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20.
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A
B
C
D
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F
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H
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P
W

ferences; then the ſpace moved over by the power will be to the ſpace moved over by the weight W, as *gfec* is to *nmld*.

Examp. If the radii of the wheels be 12 inches each, thoſe of the pinions and barrel one, the interval between two contiguous threads one; the radius AC to be 1 foot or 12 inches, its circumference will be 75 inches nearly; therefore P: W::1:129600. in caſe of equilibrio.

By this laſt theorem we may compute the increaſe of power when applied to any wheel-work, as a hand-jack, all ſorts of engines, and alſo to the block carriage uſed in artillery.

THEOREM XVI.

507. *If a power* P *be applied to the extremity* D *of a leaver, which turns an horizontal cabſtan, and the weight* W *be faſtened to the axis* AF, *by means of a rope; the power will be to the weight as the radius of the axis is to the length* FD *of the leaver, in caſe of equilibrio.* Fig. 28.

We ſuppoſe the power to be applied in a direction perpendicular to the leaver; and therefore, this machine may be conſidered as a wheel in its axis, the leaver FD repreſenting the radius.

To this machine may be referred the Gin, uſed to raiſe heavy burthens in artillery, ſuch as guns and mortars; to the drawing ſtones out of deep pits; or earth out of the ſhafts of mines; as likewiſe to the ſling-carts: The only inconveniency there is in this machine is, when the leaver is nearly vertical; then the power cannot be applied in a direction perpendicular to it; for which reaſon the vertical one, ſuch as is repreſent-

ed

ed by fig. 29, is generally preferred when it can be done.

Fig. 29. Cranes are made in this manner, as likewiſe the boring engine, and many others of that ſort; the leavers may here be made of any length, without being the leaſt burthen to the power.

As the ſcience of mechanics is the moſt extenſive, and at the ſame time the moſt uſeful part of the mathematics, it would require ſeveral volumes to explain its various uſes; for which reaſon we ſhall content ourſelves to give the principal theorems on which this ſcience chiefly depends, with a few applications to ſome ſimple machines which are moſt uſeful in artillery, referring the reader for farther enquiries of this kind to Mr. *Belidor*'s works, where he will find the greateſt variety of applications to machines that can be met with in any author whatſoever.

Of WHEEL CARRIAGES.

I have often reflected on the manner of making wheel carriages, in regard to ſome parts, which ſeem to be made quite different from what I imagined they ought to be; but as I was always cautious never to contemn ſuch practices as were univerſally received, without the cleareſt proofs of their imperfections; I found by experience that in doing ſo I was many times prevented from falling into great miſtakes. The common practice of placing the ſpokes of a wheel obliquely on the nave, which the workmen call Diſhing, ſeemed to me extraordinary; ſince if they were perpendicular, they would in appearance be ſtronger and better able to ſupport the weight on the carriage, than being oblique: I was ſo much the more

more perplexed as no author I knew had treated of it, nor the workmen themſelves could give any reaſons for it: I at laſt found the following ones.

If the ſpokes A C, inſtead of being oblique were perpendicular, as C D to the nave K L, they would certainly be in the beſt poſition, provided the carriage did always move on an horizontal plane, but when it moves on an inclined one the direction would then be oblique to that of the weight which they ſupport, and ſo would loſe their advantage of being perpendicular. And ſince when the carriage moves on an horizontal plane, each wheel ſupports but half the weight, whereas on an inclined one the loweſt ſupports ſo much the more as the plane has a greater inclination: It follows, that if the ſpokes were perpendicular to the nave, they would be oblique when one wheel is higher than the other; and therefore have the leaſt ſtrength where it ſupports the greateſt weight. Conſequently, the ſpokes are more advantageouſly placed when oblique, than if they were upright. *Fig.* 30.

Becauſe the planes on which carriages move are variouſly inclined, it was neceſſary to fix upon ſome inclination or other, as a mean between them, the workmen make them incline 3 inches in a wheel of 5 feet high; that is, if from the point C of the nave, C D be drawn perpendicular to the horizontal line A F, the part A D is three inches; from whence the angle A C D of inclination is found to be 3 degrees and 50 minutes: If A E is perpendicular to the ſpokes, it is evident that the angle E A D, which expreſſes the inclination of the plane when the ſpokes become perpendicular, is equal to the angle A C D.

Being

Being willing to know the greateſt inclination of a plane, where a carriage may juſt move upon without overſetting, I found this angle, by ſuppoſing the center of gravity 10 degrees above the center of the axletree, to be about 8 degrees and 45 minutes; which ſhews, that the angle which the ſpokes make with the nave is nearly half the angle of the greateſt inclination.

Fig. 31. The next thing to be conſidered in wheels is their height, which varies pretty much; thoſe who are for high wheels ſay, that if they meet with an obſtacle ſuch as the inclined plane L A B, they do not require ſo great a force to draw them over as leſſer ones do; but whether wheels are moved in a plane or up hill, the draught will be the ſame; ſo there is no reaſon to prefer one height before another; it is true, that when a ſtone, or ſomething of that kind, happens to be in the way, the high wheels require leſs ſtrength to draw them over than the low ones; but as this happens but ſeldom, and by accident only, it is not of ſufficient conſequence to regulate the heights of wheels thereby.

It happens ſometimes that low wheels are better than higher; as for example, thoſe of the limbers in field carriages, for the great weight of the guns lies chiefly on the wheels of the carriage, and which, by the limbers being low, is partly thrown upon them, and ſo more equally divided between the hind and the fore part of the carriage.

It has been diſputed, whether a high or a low draught is beſt, which in my opinion is eaſily decided; for it is manifeſt, that if the line C H of direction, in which the horſe draws, is parallel to the horizon, it will be the moſt advantageous of all; ſince if the wheels are higher, a part of the force is loſt by the preſſure on the ground, and when

when they are lower a part is loſt in lifting the weight upwards.

It remains now to determine the figure of the axletree; ſome will have the axis or center line of the arms to be in one continued right line; and others, that the lower parts AB, CD, ſhould be ſo: It is plain that when the lower parts AB, CD, are in a continued right line, the wheels will roll on an horizontal plane; and therefore neither endeavour to fly off and cauſe a great friction againſt the linch pins, nor to approach the body of the axletree, and cauſe a friction on that ſide; whereas if they bend upwards at the ends A, D, the wheels will roll on an inclined plane, by which they cauſe a friction againſt the linch pins, and if they either break or fall out, the wheels fly off in an inſtant, and the carriage drops down: And the contrary happens when the ends bend downwards; for then the wheels rub againſt the body of the axletree, and cauſe thereby a great friction: It is therefore neceſſary, that in all heavy carriages, ſuch as are uſed in artillery, the lower part of the arms ſhould be in one continued right line. As to the objection made by ſome, that the wheels are wider above than below, whereby they become oblique, which is not ſo well as if the arms are ſo made as that the wheels are upright; I ſay this objection is inconſiderable, in regard to the advantages which ſuch carriages have above others. In light carriages, where there is no great difference between the bigneſs of the arms, it is not neceſſary they ſhould be made ſo, the preſſures againſt the linch pins being then but ſmall. *Fig.* 32.

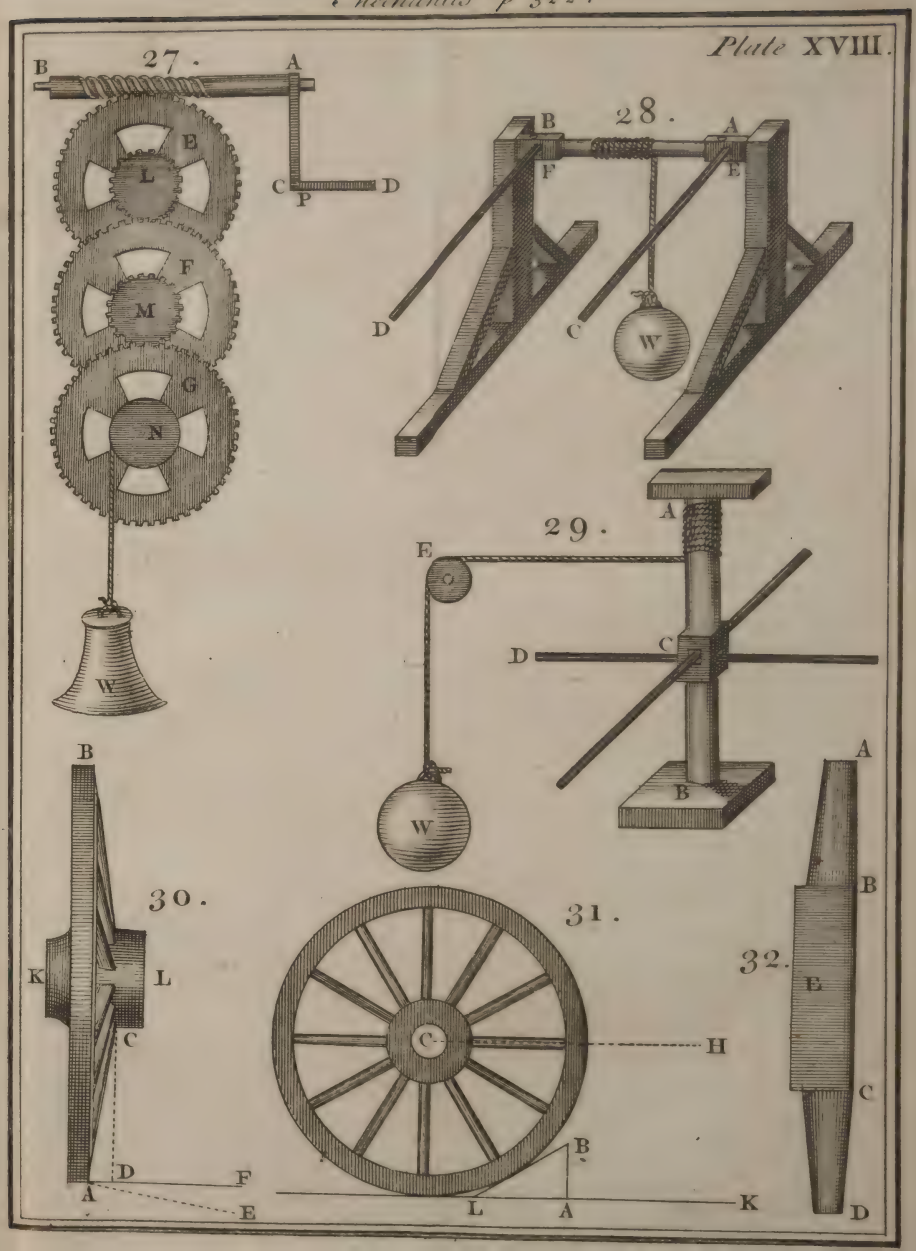
Application of MECHANICS *to the finding the Preſſure of Earth againſt Walls, and to their Strength, in order to reſiſt this Preſſure.*

The greateſt advantage that can be acquired from the ſeveral branches of the mathematics, is their application to ſome uſeful ſubject; which we can no better perform in reſpect to mechanics, than to apply it to the conſtruction of ramparts and other walls, which are made to ſupport earth, inaſmuch as there is nothing done of this kind in any *Engliſh* author I know of: Beſides, thoſe gentlemen of the royal academy of ſciences at *Paris*, who have treated on this ſubject, and after them Mr. *Belidor*, are very diffuſive in their explanations, as likewiſe very tedious in their computations; whereas with a little attention, the ſubject may be made eaſy and ſhort, as will appear in what follows.

The firſt thing wanted, is the natural ſlope that new raiſed earth would make, if cut by a vertical plane; but as earth contains various particles, ſome of which have more coheſion, others leſs, it can hardly be expected to come at any certain knowledge in reſpect to their true preſſure; all that can be done is to arrive at ſome degree of certainty, ſo as is ſufficient in practice.

It is well known that unconfined ſand will reduce itſelf almoſt to a level, and ſtiff clay will hardly form any ſlope at all, on account of the great coheſion of its parts; from whence one may give a gueſs what ſlope a middling ſort of earth would make, which is near 45 degrees, as has been found by experience, and is that which

Plate XVIII.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authors commonly supppose: But to know whether the difference of any angle, more or less, would cause any great difference in the pressure, I found that the force arising from the pressure when the slope is 40 degrees, is to the force when that angle is 50, as 446 to 453 nearly; so that the difference between those two pressures is but .07, which is inconsiderable; and therefore we may suppose the angle of the slope to be 45 degrees without any sensible error; reserving to make an addition of power when the earth is very sandy.

PROB. I.

508 *To find the Pressure of the earth* BCD, *against an obstacle* CED, *perpendicular to the base* AC; *supposing the natural slope* CB *to make an angle of* 45 *degrees with the base* AC. *Plate XIX. Fig. 33.*

It is evident that the triangle of earth CBD may be considered as a solid body sliding along the inclined plane CB, were it not retained by the obstacle CD; and therefore the weight of that body, which we call W, will be to the power which would sustain it in the direction parallel to the plane, as * the length CB of the plane is to the height; but because the triangle CAB is right angled and isoceles, CB is to AB as $\sqrt{2}$ is to unity; if therefore $\sqrt{2} : 1 :: W : \frac{W}{\sqrt{2}}$; this fourth term will express the part of the weight which is in equilibrio with the power in a direction parallel to the plane; supposing the plane perfectly smooth and polished; but by the cohesion of the particles, that power is not so much. For it has been found by experiments that a body sliding on a

* *art.* 500.

plane, middling ſmooth, requires about one third of its weight to be moved; wherefore according to this ſuppoſition, the power which ſuſtains the earth will be one third leſs than what we have found, that is, it will be expreſſed by $\frac{2W}{3\sqrt{2}}$.

Now becauſe the power in the direction parallel to the plane, is to the power in the direction parallel to the baſe CA, as CB * is to CA; that is, if we make $CB(\sqrt{2}) : CA(1) :: \frac{2W}{3\sqrt{2}} : \frac{W}{3}$,

* *art.* 501.

this fourth term will expreſs the weight which the power in the direction parallel to the baſe ſupports, or the preſſure againſt the obſtacle CED.

COR.

509. It is evident that the weight of the triangle CED may be conſidered as reunited into its center of gravity G, and the deſcent of that center, when the triangle ſlides along the inclined plane, will be that by which the deſcent of the triangle muſt be eſtimated; if, therefore, GL be drawn perpendicular to the ſide CD of the obſtacle, the whole preſſure may be conſidered as acting againſt the point L.

PROB II.

Fig. 34. 510. *Let* CDBA *be the profil of a wall; it is required to find its thickneſs* DB *above, ſo as to reſiſt the preſſure of the earth* ABE.

Draw DF perpendicular to the baſe CA, and ſuppoſe the weights Q and P to be ſuſpended in the centers of gravity N, K, of the triangle CFD, and the rectangle FCBA, and to be proportional to their reſpective areas. Now becauſe the preſſure of

of the earth ABE, endeavours to overſet the wall in the direction GL, perpendicular to AB, and the weights Q, P, retain it, in the directions perpendicular to the baſe CA; the momentum of the preſſure muſt be equal to the ſum of the momentums of the weights Q, P, in caſe of equilibrio.

If from the point C, the line CH be drawn perpendicular to the direction GL produced, of the preſſure; then as the point C is that about which the wall muſt turn, in order to be overſet, the product of the preſſure * $\frac{W}{3}$ multiplied by the diſtance CH, or its equal AL, will * expreſs its momentum; and the products of the weights Q, P, multiplied by the reſpective diſtances of their directions NQ, KP, from the point C, will expreſs their momentums. But $AL = \frac{2}{3}AB$ * and $\frac{2}{3}CF$, $CF + \frac{1}{2}AF$, expreſs theſe diſtances. Conſequently we have $\frac{2}{3}AB \times W = \frac{2}{3}CF \times Q + \overline{CF + \frac{1}{2}AF} \times P$, when theſe powers are in equilibrio.

* *art.* 508.

* *art.* 480.

* *art.* 425.

If the height of the wall AB, be a, DB or FA, x, and the baſe of the ſlope CF, na, the letter n expreſſing an indeterminate but conſtant number; then becauſe BE is equal to AB, the weight W will be $\frac{aa}{2}$; the rectangle FB or weight P, ax, and the triangle CFD, or the weight Q, $\frac{naa}{2}$; but as earth is lighter by one third than ſtone, the W $(\frac{aa}{2})$ muſt be reduced, in order to be comparable with the weights Q, P, in the

proportion of 3 to 2; which gives $W = \frac{aa}{3}$. Now if we substitute these values, into the former equation, we get $\frac{axx}{2} + naax + \frac{nna^3}{3} = \frac{2a^3}{27}$; which being divided by $\frac{a}{2}$, gives $xx + 2nax + \frac{2}{3}nnaa = \frac{4}{27}aa$; by adding $\frac{1}{3}nnaa$ to both sides, we get $xx + 2nax + nnaa = \frac{4}{27}aa + \frac{1}{3}nnaa$; the first part of this equation being a perfect square, whose root is $x + na = \frac{a}{9}\sqrt{12 + 27nn}$.

This equation is general for any slope that may be given to the wall, and becomes very simple in particular cases: For example, if the base CF is one fifth of the height AB; then will $n = \frac{1}{5}$ or .2; and the general equation becomes $x + .2a = \frac{a}{9}\sqrt{12 + 1.08} = .4a$ nearly, and $x = .2a = \frac{a.}{5}$ *Which shews that the thickness* DB *sought is always one fifth of the height* AB *of the wall, when the base of the slope is one fifth of the same height.*

And if CF is one sixth of the height AB; then $n = \frac{1}{6}$; and the general equation becomes $x + \frac{1}{6}a = \frac{a}{9}\sqrt{12 + .75} = \frac{3.57}{9}$ nearly; and so $x = .23a$. Consequently, *if the height of the wall be multiplied by* 23, *and two figures are struck off from the product as decimals, the remainder will give the thickness* BD *required.*

If the height AB is 30 feet, then 30 multiplied by .23, gives 6.9 feet.

Of

Of COUNTERFORTS.

In the building of walls which ſupport earth, buttreſſes are placed behind them, at certain diſtances from each other, in order to add more ſtrength to the wall, which muſt otherwiſe be made much thicker; theſe buttreſſes are called Counterforts; in fortification they are made three different ways, as may be ſeen in fig. 36, where their baſes are marked with the letters H, K, M, and the rectangle AF, in fig. 35, repreſents their elevation.

Mr. *de Vauban* made always the baſes of his counterforts ſuch as is marked K; and Mr. *Belidor* would have them made as that marked M; his reaſons are, that this latter ſort reſiſts better than the other, becauſe their center of gravity is farther from the wall, and are a leſſer mark to the cannon ſhot, when the wall is battered down. Here they are commonly made rectangular, as marked at H, and which we ſhall make uſe of in our computations, as being more ſimple; and their reſiſtance is but very little leſs than thoſe of Mr. *Belidor*'s; the workmanſhip is likewiſe eaſier and leſs expenſive.

Mr. *de Vauban* makes the diſtance from the *Fig.* 36. center of one counterfort to that of the next always 18 feet; but when the walls are very high they become too near each other; for which reaſon we ſuppoſe their breadth RS to be always one fourth of the diſtance from center to center, and their length ST one fourth of the height of the wall. Theſe dimenſions are eaſily adapted to any wall whatſoever; for if the diſtance from center to center is 18 feet, the breadth RS will be 4.5 feet;

if the interval is 16, that breadth will be 4 feet; the length S T depends always on the height of the wall, as we obſerved before.

PROB. III.

Fig. 35. 511. *To find the thickneſs* D B *above, of a wall with counterforts, ſo as to reſiſt the preſſure of the earth.*

From the center of gravity G of the counterfort, draw G L perpendicular to the baſe C A; then becauſe A E is $\frac{a}{4}$ by ſuppoſition, the area A F will be $\frac{1}{4}aa$, which being multiplied by C L $(x + na + \frac{1}{8}a)$ gives $\frac{1}{4}aax + \frac{1}{4}na^3 + \frac{1}{32}a^3$ for the momentum of the counterfort; but as there is an interval between them, it is too much, and therefore muſt be multiplied by the ratio $\frac{1}{4}$ between the diſtance from center to center and the thickneſs of the counterfort, which gives $\frac{1}{16}aax + \frac{1}{16}na^3 + \frac{1}{128}a^3$, for the true momentum; this being added to the momentum of the wall, found in the laſt problem, gives $\frac{1}{2}a\dot{x}x + naax + \frac{1}{16}aax + \frac{1}{3}nna^3 + \frac{1}{16}na^3 + \frac{1}{128}a^3 = \frac{2}{27}a^3$; which when divided by $\frac{1}{2}a$, will be $xx + 2nax + \frac{1}{8}ax + \frac{2}{3}nnaa + \frac{1}{8}naa + \frac{1}{64}aa = \frac{4}{27}aa$. Now if we add $\frac{1}{3}nnaa - \frac{3}{256}aa$, to each ſide, the firſt will be a perfect ſquare, whoſe root is $x + na + \frac{1}{16}a = \frac{a}{144}\sqrt{2829 + 6912nn}$.

This general equation becomes very ſhort in particular caſes: For if the ſlope is one fifth of the height A B, then $n = .2$, which gives $x + .2a + \frac{1}{16}a = \frac{a}{144}\sqrt{2829 + 276.48} = \frac{55.72a}{144}$ nearly, or $x = .12a$.

$x = .12a$. *Which ſhews that if the height of the wall be multiplied by* 12, *and two figures are ſtruck off from the product as decimals, the remainder will be the thickneſs required. If* AB = 30 *feet, then* 30 *multiplied by* .12 *gives* 3.6 *feet for* DB.

If $n = \frac{1}{6}$, then will $x + \frac{1}{6}a + \frac{1}{1}\frac{1}{6}a = \frac{a}{144}\sqrt{2829 + 921}$ $= \frac{55a}{144}$ nearly; or $x = .15a$. *Conſequently, when the baſe of the ſlope is one ſixth of the height* AB; *if this height be multiplied by* 15, *and two figures are ſtruck off from the product as decimals, the remainder will give the thickneſs* DB *required*; if AB = 30, then will DB = 4.5.

Theſe rules give a little more for the thickneſs of the wall than Mr. *Belidor* has found in his book called, *La Sciences des Ingenieurs*, book 3; the reaſon is, that he corrects a miſtake by a falſe ſuppoſition; for, in page 32, he makes the ſum of the preſſure of earth againſt a wall of 15 feet high to be 1240*b*, the letter *b* being half a ſquare foot, whereas it ſhould be 300*b* only; then to bring this number to bear he takes but half of them on account of friction, by which he finds them too little; ſo he allows one *ſixth* part of the force more to make the ſtrength of the wall ſuperior to the reſiſtance: After all theſe changes and ſuppoſitions, he finds the thickneſs of walls a ſmall matter leſs than we do.

It is true, that by diminiſhing the preſſure by one third only, as we do, for the friction, the remainder is yet too much; but as commonly one third of a weight is allowed for its friction, and the ſtrength of walls, ſhould not barely be equal to the preſſure, but above it: I preſume that

the

the thickneſs we have given here is ſufficient, and may be depended upon.

We have not hitherto mentioned any wall with parapets, which Mr. *Belidor* makes much ſtronger than thoſe who have none. As for my part I think there need be very little, and perhaps, no difference; for tho' walls with parapets ſupport a greater weight of earth than thoſe that have none; yet the latter are often charged with great weights, ſuch as guns, carriages, ſhot, and other things neceſſary in a fortification, by which they ſeem to require a greater ſtrength to ſupport theſe loads in proportion, than the weight of the ſupernumerary earth of the parapet.

It is on account of theſe reaſons that we have not given any particular computation for walls which have parapets; but thoſe who are too ſcrupulous in practice may, if they pleaſe, make an allowance of about 6 inches, or thereabouts, which will be ſufficient.

I am perſuaded that Mr. *Belidor* allows too much, for he gives a matter of 2 feet more.

PROB. IV.

Fig. 37. 512. *To find the thickneſs of a wall* DB *above, when the earth is raiſed much above the wall; ſuppoſing the ſlope of the wall to be one ſixth of the height.*

We ſhall, for conveniency ſake, ſuppoſe the ſlope BF of the earth to be parallel to that of the natual ſlope AG, though this is not always the caſe in practice; the ſmall difference between this ſlope and that commonly given, makes no ſenſible difference in the ſtrength of the wall or in the preſſure of the earth.

If we call the height BE of the earth above the wall

wall m, and the reſt as before; then the momentum of the preſſure of the triangle AEG will be * $\frac{2}{27} \times \overline{a+m}^3$; and the area of the triangle BEF, $\frac{mm}{2}$, this divided by 3 gives * $\frac{mm}{6}$ for the preſſure of that triangle in a direction perpendicular to AB, the two thirds of which $\frac{2}{3} \times \frac{mm}{6}$ or $\frac{mm}{9}$ will be that preſſure reduced to the weight of ſtone, and this multiplied by $\overline{AB + \frac{2}{3}BE}$, $(a + \frac{2m}{3})$ the diſtance of the center of gravity of the triangle, from the baſe CA, gives $\frac{3amm+2m^3}{27}$ for the momentum of that triangle; and the difference between the momentums of the triangles AEG, BEF, which is $\frac{2}{27} \times \overline{a+m}^3 - \frac{3amm+2m^3}{27}$, will be the momentum of the trapezium ABFG: This may be reduced to $\frac{a}{9} \times \overline{a+m}^2 - \frac{1}{27}a^3$.

* art. 510.

* art. 508.

Now if in the momentums of the wall and counterfort, found in the laſt problem, we ſuppoſe $n = \frac{1}{6}$, it will become $\frac{1}{2}axx + \frac{11}{48}aax + \frac{29}{172}a^3 = \frac{a}{9}\overline{a+m}^2 - \frac{a^3}{27}$; and dividing by $\frac{1}{2}a$, we ſhall have $xx + \frac{11}{24}ax + \frac{29}{172}aa = \frac{2}{9} \times \overline{a+m}^2 - \frac{2}{27}aa$; by ſubtracting $\frac{17}{612}aa$, from both ſides, the firſt will be a perfect ſquare, whoſe root is $x + \frac{11}{48}a = \sqrt{\frac{2}{9} \times \overline{a+m}^2 - \frac{229}{612}aa}$.

But

But becauſe the fraction $\frac{5}{6}\frac{2}{9}\frac{9}{1}\frac{}{2}$ is only .076 decimals, it will make but a ſmall difference in the ſquare root, and therefore may be ſafely neglected without any conſiderable error; by which the laſt equation becomes $x + \frac{1}{4}\frac{1}{8}a = \frac{a+m}{3}\sqrt{2}$; hence we get $x = .24a + .46m$ nearly. Conſequently, if in the ſum of the products of the height of the wall multiplied by 24, and that of the earth above it by 46; the two laſt figures are ſtruck off as decimals, the remainder will give the thickneſs of the wall required.

Thus if the wall is 20 feet high, and the earth 12 feet above it, the product of 20 by 24 is 480, and the product of 12 by 46 is 552; theſe two products added together give 10.32 feet for the thickneſs DB required.

Having ſhewn how to compute the thickneſſes of walls, ſo as to reſiſt the preſſure of earth, in a very eaſy and ſhort manner, we thought it would not be diſagreeable to ſome readers, eſpecially thoſe who do not underſtand algebra, to ſhew the ſame thing by a geometrical conſtruction. But before we begin, it is neceſſary to explain Mr. *de Vauban*'s general profil, repreſented by fig. 38; he makes always the thickneſs DB, near the cordon, 5 feet, and the baſe of the ſlope one fifth of the height AB; whence having the height of the wall, by adding 5 feet to one fifth of the height AB, the ſum will give the baſe CA: Therefore, if the height CS or AB, which is 80 feet, be divided into eight equal parts, the lines drawn thro' theſe diviſions parallel to the baſe, will determine the profils of walls for every 10 feet, from 10 to 80 feet high.

Fig. 38.

The length of the counterforts of a wall 10 feet high,

high is 4 feet, to which he adds conſtantly 2 feet for every 10 in height and the breadth near the wall is 3 feet, and the other 2; to the firſt he always adds 1 foot, and to the ſecond 8 inches, for every 10 feet high: But as his walls are too ſtrong when they are under 30 foot high, and too weak when they are above 40, as appears by what has been ſaid before, we ſhall ſhew how they may be made all alike ſtrong.

Becauſe the angle of the natural ſlope AE is ſuppoſed to be always 45 degrees, the triangles of earth ABE, which preſs againſt the walls, will be ſimilar; and therefore their preſſures againſt walls of different heights, will be as their heights AB. This will always hold good in the ſame kind of earth, let the angle of the natural ſlope be what it will; therefore, if the profils are made ſimilar to each other, they will be equally ſtrong, ſince the ratio of the weight of the earth to that of ſtone is conſtant; from whence ariſes this *Fig.* 34.

GENERAL CONSTRUCTION.

Let ABDC be the profil of a wall of any height, ſuch as has been found by experience to be ſtrong enough to reſiſt the preſſure of the earth; to find the profil of a wall of any other height, ſo as to reſiſt the preſſure of the earth in the ſame manner. Draw the line AD and the diagonal RB, in AB, produced if neceſſary, take BH equal to the given height, draw EHI parallel to the baſe CA, and thro' its interſections with AD and RB, the lines FL and IK, parallel to AB; then will DEFL be the profil of the wall, and BHIK that of the counterfort required. *Fig.* 39.

For

For it is evident that the figure DEFL is similar to DCAB, and BHIK to BARS.

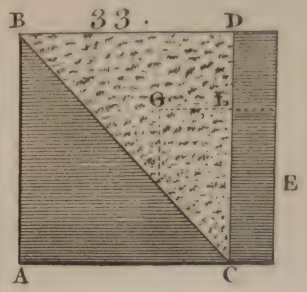
The distances between the centers of the counterforts must always be to their thickness in a constant proportion; that is, if the proportion is as 4 to unity in the given profil; it must be the same in any other.

Hence, the height AB is to the thickness BD above, in the given profil, as the given height HB or FL to the thickness LD required. And AB is to BS as HB or IK is to the length of the counterforts.

Examp. If M. *de Vauban*'s profil of a wall which is 30 feet high be supposed to have its parts in a due proportion; and we want to find the thickness above of another, which is only 20 feet high; then if we say as AB(30) is to BD(5) so is FL(20) to LD $= 3\frac{1}{3}$ feet; and because the length of his counterforts is 8 feet, if 30 : 8 :: 20 : BK $= 5\frac{1}{3}$.

And as the ratio between the distance of the center of one counterfort to that of another, and the thickness of the counterforts is one fifth nearly; the same ratio must be observed in any other profil.

SECT.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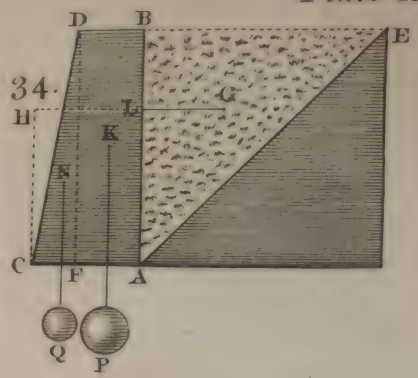
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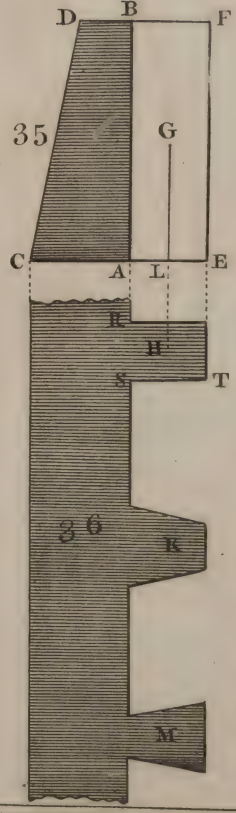
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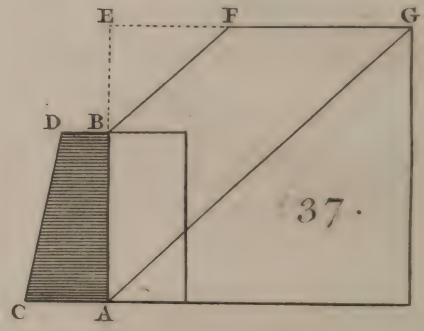
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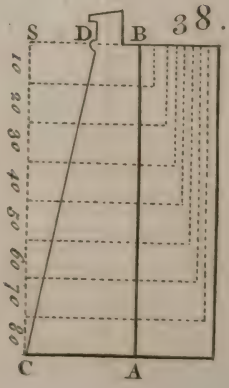
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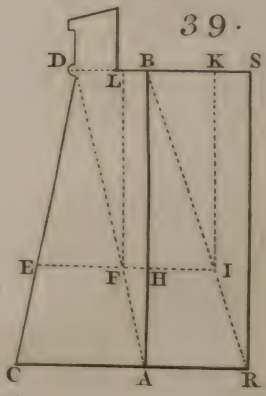
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SECT. III.

Of PROJECTILES.

THE projectiles we are going to treat of are ſuppoſed to be deſcribed in a non-reſiſting medium, tho' we know by experience that the air reſiſts conſiderably; for at *la Ferre* we made many hundreds of experiments, and always found that the ranges agreed very nearly with thoſe found by the theory, when they were ſhort, and always ſhorter in long ranges; for which reaſon the skillful bombardiers makes a competent allowance.

Many are the authors who have treated on this ſubject, who will tell you in their prefaces, that they rendered the ſubject eaſy, ſo as to be underſtood by thoſe practitioners who have no knowledge in the mathematics; but, for want of having the practice themſelves, fall generally ſhort of their promiſe.

PROB. I.

513. *If a body be projected from the point* A *in the direction* AT *with a given velocity, it is required to find the path* ABCD *deſcribed by this body.* Plate XX. Fig. 1.

If the given velocity be ſuch, that the body may deſcribe uniformly the ſpaces AG, AH, therewith, in the ſame times, it may deſcend thro' the vertical ſpaces GB, HC, by the force of gravity.

*art.440. gravity, it will describe by these joint forces, * the arc AB of a curveline: And because spaces described uniformly with the same velocity, are
*art. 433. * as the times of their description, AG will be to AH, as the time in which AG is described is to the time in which AH is described; or because AG, GB, and AH, HC, are described in the same time by supposition, AG will be to AH as the time in which the body would descend thro' the space GB by the force of gravity, to the time in which it would descend through HC by the same force: But the spaces described by falling
*art. 450. bodies, are as the * squares of the times during their fall: Therefore the square of AG is to the square of AH, as GB is to HC. Consequently, the curve ABCD is such, that by drawing lines GB, AC, TD, we have always $\overline{AG}^2$:
*art. 270. $\overline{AH}^2$ :: GB : HC. Which is the property * of the parabola.

COR. I.

514. Because the space described uniformly, with the velocity acquired by falling through GB,
*art. 451. in the same time is * double the space GB; and the spaces described uniformly in the same time
*art. 434. * as the velocities with which they are described; it follows that AG is to 2GB, as the projectile uniform velocity at A is to the accelerated velocity at B.

COR. II.

515. If AK be the height thro' which a falling body may acquire the projectile velocity; then because the spaces described by falling bodies,
*art.450 from a state of rest, are as * the squares of the velocities acquired during the fall; AK will be to GB

GB as the ſquare of the projectile velocity at A is to the ſquare of the accelerated velocity at B: But the ſquare of the velocity at A is alſo to the ſquare of the velocity at B, * as the ſquare of AG is to the ſquare of 2GB. Therefore AK:GB:: $\overline{AG}^2 : 4 \times \overline{GB}^2$, by equality of ratios; or AK: AG::AG:4GB by diviſion and multiplication. And by the ſame reaſon AK:AT::AT:4TD. *art. 514.

PROB. II.

516. *The velocity with which a body is projected* Fig. 2: *in a given direction* At, *being given, to determine the horizontal range* AB.

Erect AK, perpendicular to AB, and equal to the height thro' which a falling body may acquire the given velocity; with AK, as a diameter, deſcribe the ſemi-circumference KNA, and thro' its interſection *e*, with the given direction, draw *e*D perpendicular to AB; then if AB is made equal to 4AD, the line AB will be the range required.

For if K*e* be drawn, then becauſe AD is perpendicular to AK, the line AD touches * the circle in A; and hence the angles A*Ke*, D*Ae*, which are both meaſured by half the ſame arc A*e*, are equal; therefore the right angled triangles A*e*K and AD*e*, are ſimilar, and AK:A*e*:: A*e*:*e*D; but if Bt be drawn perpendicular to AB, and meeting the direction At in t, the ſimilar triangles AD*e*, ABt, give A*e*:*e*D::At: tB; whence we get AK:A*e*::At:tB, by equality of ratios: Now if the conſequents are multiplied by 4, and we take for 4A*e* its equal At, we ſhall have AK:At::At:4tB. Conſequently the curve paſſes thro' the * point B. *art. 111. *art. 515.

COR. I.

517. Hence, if the horizontal range AB be given, and the projectile velocity; the line of direction At, in which the body being projected with the given velocity, ſhall hit the given point B, may be found: For if AD be the fourth part of AB, and DE drawn perpendicular to AB, and thro' its interſection *e* with the circumference, a line At, then this line will be the direction required, ſince it is proved in the ſame manner as in the laſt problem that AK : At :: At : 4tB.

COR. II.

518. If the body be projected with the ſame velocity in different directions, the horizontal ranges AB will be as the ſines of angles double thoſe of elevations: For if the radius ON be drawn perpendicular to the diameter AK, interſecting D*e* produced in L; then will OL, or its equal AD, be the ſine of the arc A*e*, which meaſures an angle double the angle *e*AD of elevation; and ſince AD is always the fourth part of the horizontal range AB, it is evident when AK remains the ſame, the ſines AD of angles double thoſe of elevations, will always be in the ſame proportion to one another as the horizontal ranges.

COR. III.

519. If the line D*e* interſects the circumference in two points *e*, E, it is manifeſt that there are two directions At, AT, in which the body being projected with the ſame velocity, acquired by falling thro' KA, will hit the ſame point B in the horizontal line AB: For the triangles AEK, ADE,

ADE, are ſimilar; and therefore AK:AE:: AE:ED, and AK:AE::4AD:4ED; or AK:4AE::4AE:16ED: But if the line AT was produced, ſo as to meet Bt produced, it would be equal to 4AE; becauſe AB is 4AD, and Bt produced 4DE. Conſequently the body would hit the point B.

COR. IV.

520. Hence, becauſe DE is parallel to AK, the arcs A*e*, KE, are equal; it is evident that if the direction in which the body is projected, makes the ſame angle either with the vertical line AK, or with the horizontal one AB, that body will have the ſame horizontal range.

COR. V.

521. If the line DE inſtead of cutting the circle only touches it in N, the angle NAB of elevation being then meaſured by half the quadrant A*e*N, will be 45 degrees; and the horizontal range, which is 4AD, will become four times the radius, or twice the height AK, and is the greateſt that can be attained with the ſame projectile force.

COR. VI.

522. If the arc A*e*, or KE be 30 degrees, its ſine OL, or AD will be equal to half the radius AO; and the horizontal range AB, which is quadruple of AD, will be equal to the diameter AK. Conſequently, when the angle of elevation is 15 degrees, the horizontal range is then juſt equal to the height thro' which a falling body acquires the projectile velocity; or half the greateſt

*art. 521. range * that can be attained by the body with the ſame projectile force.

PROB. III.

523. *If a body be projected in a given direction* A t, *with a given velocity, to find the higheſt point to which that body will aſcend.*

Suppoſe the force of projection in the direction A *e* to be reduced into the horizontal one A D, and into the vertical one D *e*; ſo that by the firſt A D, it is carried from A to B; and by the ſecond D *e* it riſes upwards; but as the force of gravity conſtantly acts in a contrary direction to the force D *e*, it will in equal times deſtroy equal parts of that force, and the body will riſe upwards till the whole force in that direction is deſtroyed; then it will deſcend by the force of gravity till it arrives at B, where it will have acquired a force equal to that which was deſtroyed; therefore the aſcent and deſcent thro' the ſame vertical ſpace being made in equal times, the body will arrive to the higheſt point in half the time that the projectile is deſcribed; but becauſe a body would fall through the *art. 513. ſpace B t, in the ſame * time that the projectile is deſcribed, it will fall thro' one fourth of that ſpace in half the time; ſince the ſpaces deſcribed by falling * art. 450. bodies from a ſtate of reſt are as the * ſquares of the times, and the ſide of a ſquare double another ſquare, is quadruple the ſide of the other; but be- * art. 516. cauſe A D is one fourth of * A B, D *e* will alſo be one fourth of B t. Conſequently D *e* is the greateſt height to which the body aſcends in deſcribing the projectile.

COR. I.

524. It is evident that D *e* is the verſed ſine of the

the arc A*e*; and becauſe the altitude of the projectile is equal to that ſine, it follows that the altitudes of projectiles deſcribed with the ſame force, are to one another as the verſed ſines of arcs double thoſe which meaſure the angles of elevation.

COR. II.

525. Since ſpaces deſcribed by falling bodies from a ſtate of reſt are as the ſquares of the times * in which they are deſcribed; the time in which a body would fall thro AK, is to the time in which a body would fall thro' *e*D, as AK is to A*e*: For ſince the lines AK, A*e*, and *e*D, are * in a continual proportion, AK is to D*e* as the ſquare of AK is to * the ſquare of A*e*; and as the ſquare of the time of the body falling thro' AK is to the ſquare of the time of the body falling thro' *e*D, as AK is to D*e*, or as the ſquare of AK to the ſquare of A*e* by equality of ratios. Conſequently, the time in which a body would fall thro' AK, is to the time in which a body would fall thro' *e*D, as AK is to A*e*.

* *art.* 450.
* *art.* 516.
* *art.* 76.

COR. III.

526. Becauſe the time in which the body deſcribes the projectile is double the time * in which the body would fall thro' *e*D; it is evident, that the time a body would fall thro' AK, is to half the time in which the projectile is deſcribed as AK is to A*e*. Therefore the times in which projectiles are deſcribed with the ſame force, are to each other as the chords A*e* of angles double thoſe of elevations; or becauſe theſe chords are as the ſines of half the angles which meaſure the

* *art.* 523.

arcs; the times will be as the ſines of the angles of elevations.

COR. IV.

527. It is alſo manifeſt, that if a body be projected under an angle of elevation of 45 degrees, with different forces, the horizontal ranges will be to each other as the ſquares of the times in which they are deſcribed: For when D*e* only touches the circle in N, D*e* and AD become each equal to the radius AO; and the horizontal range AB is then equal to four times the radius; and ſince the line *e*D is deſcribed in half the time that the projectile is deſcribed, a body would fall through a height double the height AK in the * ſame time that the projectile is deſcribed; that is, it would fall thro' a height equal to the horizontal range AK. Conſequently, the horizontal ranges deſcribed with an angle of 45 degrees are as the ſquares of the times.

* *art.* 450.

PROB. IV.

Fig. 3, 4. 528. *The inclination of a plane* AB *either above or below the horizon* AP, *the line of direction* At, *and the projectile force being given; to find the point* B *where the body will hit that plane.*

On the horizontal line AP erect AK perpendicular, and equal to the height through which a falling body may acquire the given velocity; biſect the line AK by the perpendicular ON, and draw AO perpendicular to the inclined plane AB; then if from the center O, an arc of a circle be deſcribed through the points A, K, and through its interſection *e*, with the given direction, *e*D perpendicular to AP; by taking AP quadruple to AD, the perpendicular PB to AP will determine the point B required.

For

For becauſe AB is perpendicular to the radius AO, it will be a tangent to the circle; and ſo the angle BA*e* is equal to the angle AK*e*, as well as the alternate ones KA*e*, A*e*D; ſo that the triangles AK*e*, A*e*F, are ſimilar, and therefore AK:A*e*::A*e*:*e*F; but the parallels F*e*, Bt, give A*e*:*e*F::At:tB; whence, by equality of ratios we have AK:A*e*::At:tB; and ſince AP is the quadruple of AD, AB will be the quadruple of AF, and At quadruple of A*e*; if therefore we take the quadruples of the conſequents we ſhall have AK:4A*e* or At::At:4Bt. Conſequently, the projectile * paſſes thro' the point B. * *art.* 515.

COR. I.

529. Hence, if the range AB, the inclination of the plane and the projectile velocity, are given; the line of direction at At will be found by taking AD equal to one fourth of AP, or AF to one fourth of AB; by drawing D*e* parallel to AK, and At through its interſection *e* with the arc ANK.

COR II.

530. It is evident that when the line D*e* interſects the arc in two points, there will be two directions At, AT, in which the body being projected with the ſame velocity, ſhall hit the ſame point B in the inclined plane; and when that line only touches the arc in N, the range will then be the greateſt poſſible that can be attained with the ſame velocity.

COR. III.

531. Becauſe AK is perpendicular to AP, and AO to AB, the angle OAK is equal to the an-

gle of inclination P A B of the plane; and the angle N O A its complement. Conſequently, the angle of elevation, which correſponds to the greateſt range, is half the complement to the angle of inclination of the plane, when the plane is above the horizon; or to half the ſupplement (fig. 3) of the complement, when the plane is below the horizon.

PROB. V.

532. *The point* B *on a plane* A B *whoſe inclination is given, together with the projectile velocity being given; to find the angle of elevation* D A *e ſuch as the body ſhall hit the given point* B.

Becauſe the angle B A P, or its equal O A G, is given as well as A G, being half the height A K, thro' which a falling body acquires the given velocity; therefore as the radius is to the tangent of the angle O A G, ſo is A G to G O; and as the radius is to the ſecant of that angle, ſo is A G to A O or O *e*; again, as the radius is to the coſine of the ſame angle, ſo is A F the fourth part of the given diſtance A B to A D, or its equal G L, to which add O G in fig. 4, or ſubtract it in fig. 3; then the ſum or difference O L will be the coſine of the angle N O *e*, which is the difference between the given angle A O N and the angle A O *e*, or F A *e*, to which the angle D A *e* beeig adedd in fig. 3, or ſubtracted in fig. 4, gives the angle D A *e* of elevation; which conſequently is known.

PRACTICAL RULES *for* HORIZONTAL RANGES.

I. *The range of a body projected with an angle of* 15 *degrees is half the range of that body, if pro-*

projected * *with the same force under an angle of 45 degrees.* * art. 522.

II. *The range of a body projected with an angle of 45 degrees, is equal to the square of the time of its description expressed in seconds, multiplied by 16.1 feet.*

This rule results from the law of falling bodies, which describe spaces proportional to the squares of the times; and a body falling thro' the space of 16.1 in a second, and that the time of description of the projectile is * equal to the time of a falling body describing a space equal to the horizontal range. * art. 527.

III. *If a body be projected with the same force, under different angles of elevations; the horizontal ranges are as the sines * of angles double those of elevations respectively.* * art. 518.

Examp. I. *Let a body projected under an angle of 45 degrees be 12 seconds in its flight: What is the horizontal range.*

The square of 12 is 144, which multiplied by 16.1 feet gives 2318.4 feet, or 772.8 yards, by rule the second, for the range required.

Examp. II. *If the range of a body projected under an angle of 25 degrees be 200 yards, what will be the range if the body be projected with the same force, under an angle of 30 degrees?*

The sine of 50 degrees, double of 25, is 76604, and the sine of 60, double that of 30, is 86602; therefore 76604 : 86602 :: 200 : 226 = to the range required by the third rule.

Examp. III. *If the range of a body projected with*

with an angle of 20 *degrees be* 200 *yards; what muſt the angle of elevation be to project the body with the ſame force at a diſtance of* 300 *yards?*

The ſine of 40 degrees, double of 20, is 64278; whence 200 : 300 :: 64278 : 96417 = to the ſine of an angle double the angle ſought; this ſine anſwers to an angle of 74 degrees and 37 minutes; half of which 37 degrees and 18.5 minutes, will be the angle required.

The ſecond rule is uſeful at ſea, where no trial of the force of the powder can be made; it will be ſufficient to know the diſtance from the ſhip to the object, then the time required for the ſhell to hit the object, muſt be found by ſaying, 16.1 feet is to the given diſtance as the ſquare of one ſecond is to the ſquare of the time required, and extracting the ſquare root, which gives the time, ſo that when the time of the flight of the ſhell is obſerved, you will ſee whether it exceeds or falls ſhort of the time found, and the range will exceed or fall ſhort of the given range. Thoſe who are unacquainted with the practice, may imagine that one might ſee whether the ſhell falls in the place aimed at, and therefore the rule is uſeleſs; but in practice it is otherwiſe; for a ſhip may be at leaſt a mile off from the object, and the byſtander fancy that the ſhell did fall very near the place, when it may be 4 or 500 yards from it.

If it is propoſed to throw a ſhell at a given diſtance, it will be moſt convenient to throw a ſhell under an angle of 25 degrees of elevation, then if twice the range is either equal to or greater than the given diſtance; that quantity of powder will be ſufficient to throw the ſhell to the given diſtance, if not, another trial muſt be made with more powder; whereas if the firſt ſhell is thrown

under

under any other angle of elevation, it is not ſo eaſily known whether that quantity of powder is ſufficient to throw the ſhell to the diſtance propoſed.

It would be very uſeful to know the greateſt ranges of all the different mortars, loaded with different charges; for having a table of theſe ranges one might eaſily know what quantity of powder is ſufficient to throw the ſhell upon an occaſion to a given diſtance.

N. B. Theſe are all the practical rules uſeful in gunnery, which are but few and plain; from whence it might be imagined that a ſhell may be thrown ſo as to hit any propoſed object, without any difficulties; but it muſt be conſidered that there are many unavoidable accidents in practice, which renders it very difficult; ſuch as the inequality of the powder, unequally placed in the chamber, the mortar or bed not being truly made; the platform giving way in one place or other; the inequality of the weight and roundneſs of the ſhell; to this we may add the reſiſtance of the air, which is very great, and varies the ranges in proportion as they are long. However, the practitioner muſt not conclude from thence, that theſe rules are of no ſervice, ſince there are ſo many obſtacles which prevent their exactneſs; for tho' it is impoſſible to attain to a great exactneſs, yet notwithſtanding, by taking all the care that a ſkillful bombardier is capable of, they will anſwer ſufficiently in moſt caſes, as I have found by experience.

We have not given any practical rules for throwing ſhells on inclined planes, becauſe the theory is ſo difficult as the practitioners will never follow.

As no author I know of has ſolved the problems

lems of gunnery, when the resistance of the air is considered, we shall insert them here without any demonstration, because they depend on higher principles than we propose to treat of here in this work.

PROB. VI.

Fig. 2. 533. *The angle of elevation being given, it is required to find the abscissa* A C, *which corresponds to the greatest height* C M, *of the projectile, in a medium whose resistance is as the square of the velocity.*

Let a be the tangent of the angle of elevation; s the secant; p the height through which a falling body may acquire the projectile velocity; and e the height thro' which a falling body may acquire the greatest velocity in that medium; then will

$$\text{A C} = \frac{2ap}{ss} - \frac{4aapp}{es^3} + \frac{4ppa^4}{3es^4} + \&c. \text{ nearly.}$$

$$\text{And C M} = \frac{aap}{ss} - \frac{4a^3pp}{3es^3} + \frac{2a^5pp}{3es^5} + \&c.$$

PROB. VII.

534. *The same thing being supposed as before, to find the range* A B.

$$\text{We shall have A B} = \frac{4ap}{ss} - \frac{32aapp}{3es^4} + \frac{16a^4pp}{3es^5} + \&c.$$

PROB. VIII.

535. *It is required to find the cosine u of the angle of elevation which gives the greatest range.*

We

We ſhall have $uu = \frac{96e - 235p}{192e - 512p}$ *proximae.*

Hence the coſine u is greater than $\sqrt{\frac{1}{2}}$, and conſequently the angle of elevation which gives the greateſt range, is leſs than 45 degrees.

It may be obſerved that the angle of elevation which gives the greateſt range is not always the ſame, but changes according as the projectile velocity is more or leſs.

PROB. IX.

536. *The horizontal range* AB (a) *being given, as well as the projectil velocity* (p) *to find the ſine* z, *of the angle of elevation.*

Suppoſing A to expreſs that ſine in a non reſiſting medium, we ſhall have $z = A + \frac{aa}{6pe} + \frac{a^4}{192ep^3} + \&c.$ *proximae.*

PROB. X.

537. *The range* AB *of a body projected under an angle of* 45 *degrees being given, to find the parameter* p *of the curve.*

We ſhall have $p = \frac{e\sqrt{2}}{4} + \sqrt{\frac{ee}{8} - \frac{ea\sqrt{2}}{4}}$; or,

$p = \frac{a}{2} + \frac{aa}{2e\sqrt{2}} - \&c.$ *proximae.*

If the angle of elevation be 30 degrees, and the horizontal range b; then will $p = \frac{b}{\sqrt{3}} + \frac{7bb}{18e} \&c.$ *proximae.*

PROB.

PROB. XI.

538. *The two ranges a, b, in the laſt problem, being ſuppoſed to be known, it is required to find the greateſt velocity e, which the body can acquire in that medium.*

If we ſuppoſe $m = \frac{a}{2} - \frac{b}{\sqrt{3}}$, $n = \frac{7bb}{18} - \frac{aa}{2\sqrt{2}}$, and $r = \frac{25a^3}{72} - \frac{2b^3}{27\sqrt{3}}$; then will $e = \frac{n}{2m} + \sqrt{\frac{r}{m} + \frac{nn}{4mm}}$, *proximae*, or $\frac{1}{e} = -\frac{n}{2r} + \sqrt{\frac{m}{r} + \frac{nn}{4rr}}$

When the reſiſtance is nothing, the velocity e becomes infinite in reſpect to the other quantities; and therefore, by ſuppoſing all the terms which are divided by e nothing in the preceding problems, we ſhall have thoſe expreſſions for a non-reſiſting medium.

REMARK.

In order to apply the preceding theory to practice, it is neceſſary to know whether the reſiſtance of the air is really as the ſquares of the velocities, which ſeems to be the caſe in bodies projected with ſuch great velocities as ſhells are, tho' it may not be ſo when the projectile velocity is but ſmall; the next thing to be found is to know, whether the reſiſtance does not ſenſibly alter in wet or dry, cold or warm weather; if the reſiſtance is ſuch as we have ſuppoſed, and it does not ſenſibly change in different weather, the problems above may be eaſily applied to practice.

If two experiments were made at a time in order to find the quantity e, and they were repeated

in

Plate XX.

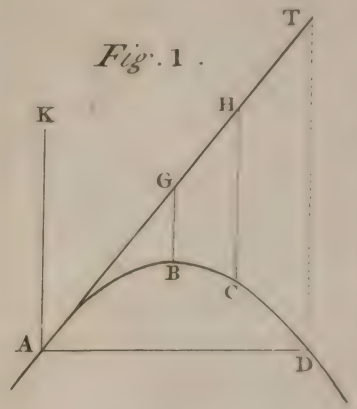
Fig. 1.
T
K
H
G
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C
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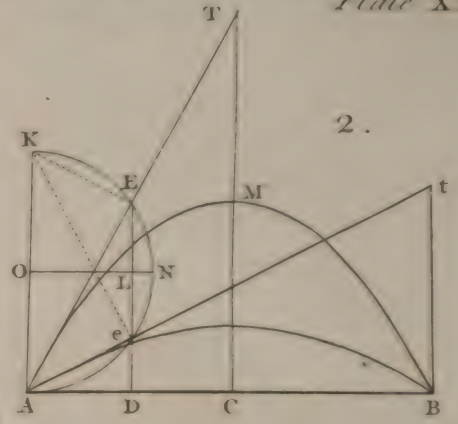
D


2.
T
K
E
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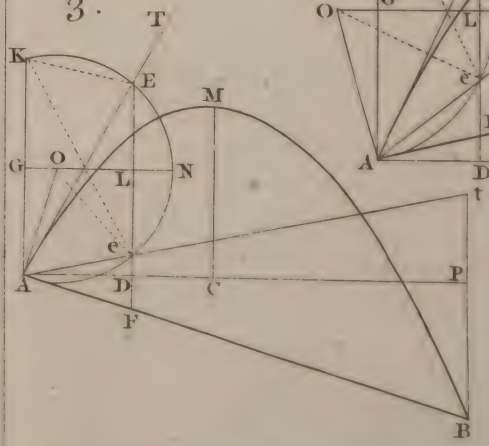
3.
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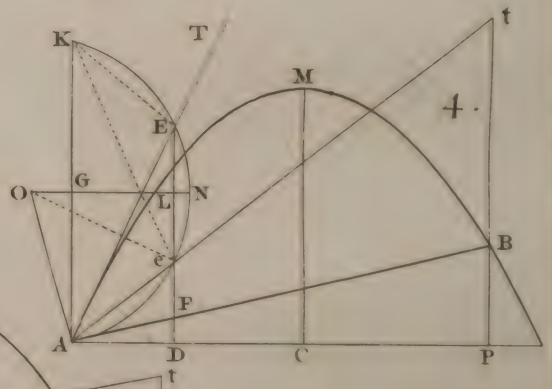
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in different ſeaſons, one might know whether theſe ſuppoſitions are well grounded; but if the values of e, found in different ſeaſons, vary much, I queſtion whether there can ever any better theory be found, than that which reſults from the properties of the parabola.

Sir *Iſaac Newton* has proved in prop. 40, of the ſecond book of his principia, that the greateſt velocity which a body can acquire in a reſiſting medium, is equal to the velocity it can acquire in a non-reſiſting medium, when it deſcribes a ſpace, in its fall, which is to four thirds of its diameter, as the denſity of the globe is to the denſity of the fluid. Hence, if e be that ſpace, d the diameter of the globe, and m to n as the denſity of the globe to the denſity of the medium; then will $e : \frac{4d}{3} :: m : n$; or, $e = \frac{4md}{3n}$: Now becauſe the denſity of caſt iron is to that of the air as 7425 to $\frac{1}{4}$; or as 5940 to unity; if we make $m = 5940$, and $n = 1$, we ſhall have $e = 7920d$. Which ſhews that the greateſt velocity a cannon ball can acquire in the air, is equal to that which a falling body would acquire by falling thro' a ſpace 7920 times its diameter, in a non-reſiſting medium. As ſhells are hollow their ſpecific gravity will be leſs than that of the ball; and therefore the number which expreſſes the height thro' which a falling body may acquire a velocity in a non-reſiſting medium, equal to the greateſt velocity that a ſhell can acquire, is leſs than that above.

SECT.

SECT. IV.

METHOD *for finding the greateſt or leaſt of* QUANTITIES.

THIS method is very uſeful in mechanics for making all ſorts of machines, ſo as to produce the greateſt effects poſſible; but as the principles on which it depends are generally demonſtrated by fluxions, which few practical people underſtand, it has not hitherto been ſo much applied as it ſhould; for which reaſon we ſhall here deduce the moſt uſeful problems from common geometry, referring the reader for the reſt to my treatiſe of conic-ſections, where this method has been treated to its full extent.

PROB. I.

Plate XXI. *Fig.* 1.

539. *To inſcribe the greateſt parallelogram* EFGH *poſſible, in a given quadrilateral* ABCD.

Biſect any ſide AB in F, draw FG, FE, parallel to the diagonals AC, BD, and from the interſection E of the latter with the ſide AD; EH, parallel to AC; by joining the points G and H, the figure EG will be the parallelogram required.

For becauſe FG is parallel to AC and biſects
* *art.* 67. AB; it will *alſo biſect BC; by the ſame reaſon EF biſects AD, and EH, DC. Therefore, ſince the ſides BC, DC, are biſected in G and
art. 68. H, the line HG will *be parallel to DB or EF. Conſequently EFGH is a parallelogram.

Now

Now if from any point *f* in A B the lines q l, *f* p, are drawn parallel to A C and D B, meeting D B in q, and A C in p; and from the point A, a line parallel to D B, meeting Q F, q*f*, produced in L, l; then becauſe the complements F p, F l, of the parallelogram p l, are * equal; * *art.* 38.
and the parallelograms F l, *f* Q, being between the ſame parallels, are as their baſes * F L, *f* q. * *art.* 65.
But F L is equal to F Q, by conſtruction, and *f* q is leſs than F Q, when the point *f* falls between F and B: Therefore the parallelogram *f* Q is leſs than the parallelogram F l, or its equal F p; and the parallelogram p q leſs than the parallelogram P Q.

If the point *f* falls between the points A and F, it may be proved in the ſame manner, that the part cut off from the parallelogram P Q will always be greater than the part added; and ſince what is true in regard to the part P Q of the parallelogram E G, will be ſo in regard to the whole; it follows that the parallelogram E G is the greateſt which can poſſibly be inſcribed in the quadrilateral A B C D.

It is plain, that the inſcribed parallelogram is half the quadrilateral; and that if the ſides of any triangle are biſected by a line, and others are drawn from theſe interſections perpendicular to the baſe, they will make the greateſt rectangle that can be inſcribed in the triangle.

If a point either within or without a quadrilateral be required, ſuch as the ſum of the lines drawn from this point to the four angles, ſhall be the leaſt poſſible, it is manifeſt that the point of interſection O of the diagonals, will be the required one: For if you imagine lines to be drawn from any other point z, to the angles, the ſum of

Az and Cz will be greater than AC, and the sum of Bz and Dz greater than DB. Consequently the sum of the diagonals is less than the sum of the lines drawn from any point z, to the four angles.

PROB. II.

Fig. 2, 3. 540. *If in the axis or the tangent at the vertex of any curve* AM, *the line* AQ *be taken of any given length, and* QN *drawn perpendicular to it; it is required to inscribe the greatest rectangle possible in the given space* AQN.

Let the tangent MT be drawn in such a manner, that by drawing from the point of contact MP perpendicular to AQ, the subtangent PT be equal to PQ; then if ML is parallel to AQ, the rectangle PMLQ will be that required.

For if from the point T, where the tangent meets the line AQ, there be drawn a line parallel to PM, meeting LM produced in D, and from any other point m, in the tangent ld parallel, and mp perpendicular to AQ; then because the complements Mp, Md, of the parallelogram pd, are equal, and the parallelograms Md, mL, be-
* *art.* 65. ing between the same parallels, * are as their bases MD, ml; and since ML is equal to MD by construction, ml is less than ML. Therefore the parallelogram mL is less than the parallelogram Md, or its equal Mp; and the parallelogram pl less than the parallelogram PL.

Of which side soever the point m is taken in regard to the point M, it may be proved that the part cut off from the rectangle PL is always greater than the part added. Consequently, the rectangle PL is the greatest that possibly can be inscribed in the figure AMNQ.

As

As the point m may be taken ſo near to the point M that there is no aſſignable difference between the line p m, and the ordinate in that place, it is evident that the foregoing demonſtration holds good, whether that point be in the tangent or in the curve: Beſides, in fig. 2, when p M remains the ſame, and the point m be ſuppoſed in the curve, the part m L will ſtill be leſs than it was before.

EXAMPLE I.

541. Let A M N be the common parabola, *Fig.* 2. then the ſubtangent P T, or its equal P Q, will be * double the abſciſſa A P; and therefore A P * *art.* 261. is one third of the given line A Q. Hence, if A P $= x$, P M $= y$, and p the parameter of the axis; we have * $px = yy$ by the propriety of * *art* 256. the parabola; and if A Q $= a$, then will P Q $= a - x$, and the rectangle P L will be $ay - xy$; or, if inſtead of x we ſubſtitute its equal $\frac{yy}{p}$; that rectangle will be $\overline{ap - yy} \times y$, by neglecting p in the denominator; which will be the greateſt poſſible when $3x = a$, or $ap = 3yy$.

EXAMPLE II.

542. Let A M N be again the common parabola, and the tangent M T to meet the axis in R, and L D produced in H. Then becauſe A R is equal to A H or to P M, the triangles R A T, M P T, are ſimilar and equal; and A T $=$ T P, equal to P Q by ſuppoſition. Therefore P Q is one third of A Q. *Fig.* 3.

If A Q $= a$, A H $= x$, H M $= y$, and p the parameter, then will P Q $= a - y$, and the rect-

angle P L, equal to $\overline{a-y} \times x$, or by substituting yy, for x; $ayy-y^3$, which is the greatest possible, when $3y=2a$.

These two examples serve to solve almost all the useful mechanical problems of this kind, as will appear hereafter.

EXAMPLE III.

Fig 4. 543. Let A M B be a quadrant of a circle, or an ellipsis, described from the center C, then because we have, by the property of these curves,
*art.126. * C P : C A :: C A : C T = 2 C P by supposition, or $2 \times \overline{CP}^2 = \overline{CA}^2$.

If the radius C A $= a$, C P $= x$; then will P M $= \sqrt{aa-xx}$, in the circle; and the rectangle P L will be $x\sqrt{aa-xx}$, which is the greatest possible when $2xx=aa$.

EXAMPLE IV.

Fig. 5. 544. Let A M N be an equilateral hyperbola, whose semi-axis C A is $=a$, C P $=x$, and if C Q $=d$; then by the property of the hyperbola
*art.334. we * have C P : C A :: C A : C T $= \frac{aa}{x}$, and C P $-$ C T $=$ T P or P Q; that is, $x-\frac{aa}{x}=d-x$.

Hence, $2xx-dx=aa$; the square root of which gives $x=\frac{1}{4}d+\sqrt{\frac{1}{2}aa+\frac{1}{16}dd}$.

*art. 325. Because P Q $= d-x$, and * P M $=\sqrt{xx-aa}$; the rectangle P L will be $\overline{d-x} \times \sqrt{xx-aa}$; and the greatest possible, when x is equal to the expression above.

EXAMPLE V.

545. Let BQ, RQ, be drawn perpendicular to the aſymptotes CB, CR, of the equilateral hyperbola AMN; and let the tangent Mt, produced, meet QB in T; then becauſe of the ſimilar triangles TPM, MDt, we have TP or PQ : DM :: PM : Dt = DC, * by the proper- *art 338.
ty of the hyperbola; And by compoſition, BQ : DM :: CB : CD; but we have alſo * CR : *art. 350.
DM :: CD : RN; and becauſe BQ is equal to CR, we have CB : CD :: CD : RN, by equality of ratios. Fig. 6.

If CD = x, MD = y, CB = c, CR = b, RN = a; then the laſt proportion gives $c : x :: x : a$, or $ac = xx$; and becauſe ML = $b - y$, and MP = $c - x$, the rectangle PL will be $\overline{b-y} \times \overline{c-x}$; or becauſe * $ab = xy$, this rectangle becomes *art. 350.
$\frac{b}{x} \times \overline{c-x} \times \overline{x-a}$; which therefore is the greateſt of all, when $xx = ac$.

It is evident from the foregoing examples, that the method of finding the greateſt and the leaſt of quantities, depends on that of drawing tangents; and in whatever figure, where the proportion between the abſciſſa and the ſubtangent is known, the greateſt rectangle that can be inſcribed in a given ſpace of that figure, will alſo be known; thoſe we have given already are ſufficient for our purpoſe; we ſhall only add a few problems, in order to ſhew the application of what has been ſaid on this ſubject.

PROB. III.

546. *To divide a given line* AB, *in* E, *ſo as when biſected in* C, *the product or ſolid* AE × EB × Fig. 7.

$EB \times CE$, *shall be the greateſt poſſible of all the like products.*

Fig. 2. In the axis AQ of the parabola, whoſe parameter is unity, take AQ, a third continued proportional to the parameter, and to half CB the given line; then if AP is one third of AQ, and CE is taken equal to the ordinate PM, the point E will be the required one.

For becauſe $1 : CB :: CB : AQ$, we have $QN = CB$, by the property of the parabola; and the difference between the ſquares of QN, PM, or their equals CB, CE, is equal to PQ; and ſince $PM = CE$ by conſtruction, the rectangle QM will be equal to the difference between the ſquares of CB and CE, multiplied by CE; or becauſe the rectangle $AE \times EB$, is equal to the difference between the ſquares of CB, CE, the rectangle QM will be equal to $AE \times EB \times EC$. Which conſequently is the greateſt of all when AP is one third of AQ, or

* *art.* 541. when* $3EC^2 = \overline{CB}^2$.

PROB. IV.

Fig. 8. 547. *To inſcribe the greateſt cone poſſible* CNM *in a ſemi-ſphere.*

Draw the diameter AB, in which take the point P, ſuch that the ſquare of CP be one third of the ſquare of the radius; then the line NM, drawn at right angles to AB, will be the diameter of the baſe of the cone required.

For becauſe the areas of circles are as the ſquares of their radii, the ſquare of PM, or the rectangle $AP \times PB$ its equal, will be as the baſe of the cone, and $AP \times PB \times CP$ as the cone itſelf; but this product is, by the laſt, the greateſt of

all

all when the ſquare of C P is one third of the ſquare of C B; therefore the conſtruction is true.

COR. I.

548. If the line A B is required to be divided into two ſuch parts, that their product ſhall be the greateſt poſſible; the point C which biſects that line will be the required one. For becauſe the ſquare of P M being always equal to the rectangle A P × P B; and is the greateſt of all when N M paſſes through the center C.

COR. II.

549. It is manifeſt, that of all the rectangles which have the ſame area, the ſquare has the leaſt boundary. For ſince the ſquare of P M is always equal to the rectangle A P × P B; and the ſum of two ſides N M, is always leſs than the ſum A B of two ſides of the rectangle, and 2 N M leſs than 2 A B, by the property of the circle; the ſquare has the leaſt boundary of all equal rectangles.

This corollary ſhews that the outſide walls of a ſquare building are leſs than thoſe of any other rectangular figure of the ſame area.

PROB. V.

550. *To inſcribe the greateſt cylinder* N H G M *poſſible, in a ſphere.*

If the ſquare of C P be one third of the ſquare of C A, then N M will be the diameter of the baſe of the cylinder wanted,

For this cylinder will be as $2\,C\,P \times \overline{P\,M}^2$, or as its equal A P × P B × C P, and this product has been proved * to be the greateſt of all, when the ſquare of C P is one third of the ſquare of C A. * art. 546

It muſt be obſerved, that whether a quantity be multiplied or divided by a conſtant quantity, its greateſt or leaſt is not changed thereby; and likewiſe, that if the ſquare, or any root or power of a quantity is the greateſt quantity poſſible, the quantity itſelf will be likewiſe the greateſt poſſible; this is plain from the nature of figures; ſince if the ſide of a ſquare or cube is greater than that of another, the ſquare or cube will be ſo too.

PROB. VI

Fig. 9. 551. *To divide a line into two ſuch parts, that the ſquare of the one multiplied by that of the other, ſhall be the greateſt poſſible.*

Let the given line AB be divided into three equal parts, ſo that CB be one third of it; then the product of the ſquare of AC, multiplied by CB, will be the greateſt poſſible.

For if $AB = a$, $AC = y$; then will $BC = a - y$; and $\overline{a-y} \times yy$, or $ayy - y^3$, expreſs the given product, which, by art. 542, is the greateſt of all when $3y = 2a$.

COR.

Fig. 10 552. Hence, if the diameter AB of a circle be divided at P, ſuch that AP be one third of it; the line NM drawn through the point P at right angles to the diameter, will be the diameter of the baſe of the greateſt cone poſſible, NBM, that can be inſcribed in a ſphere; ſince it is as $AP \times \overline{PB}^2$.

PROB. VII.

553. *To find the greateſt poſſible of a fraction, ſuch as* $\frac{x}{\overline{a+x} \times \overline{c+x}}$.

The

Plate XXI

Fig. 1.
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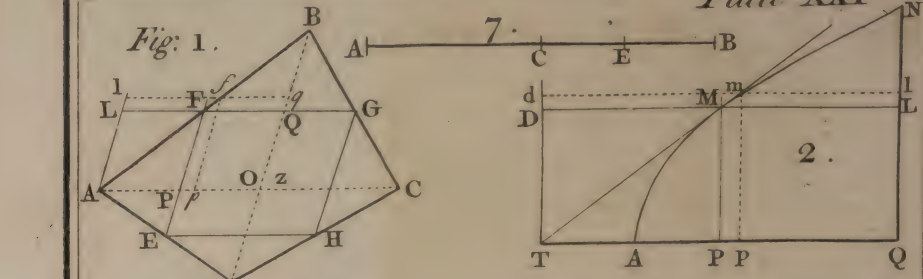
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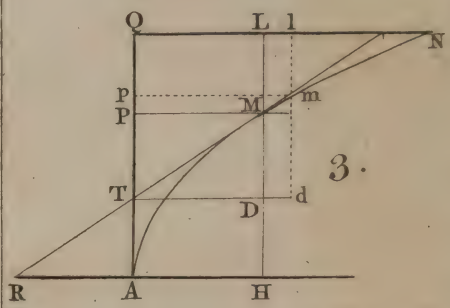
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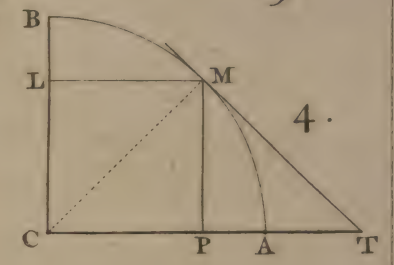
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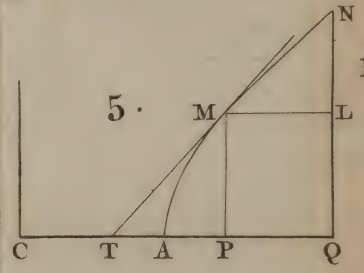
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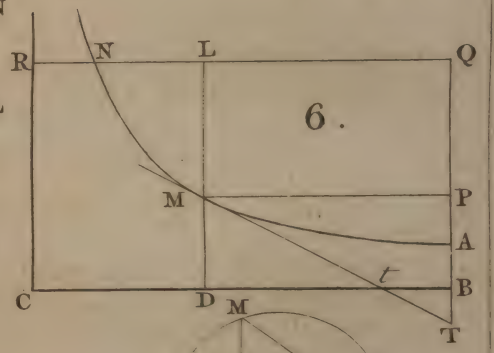
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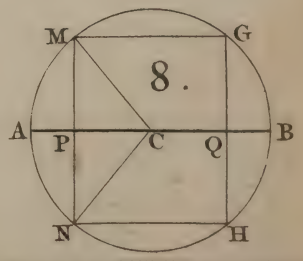
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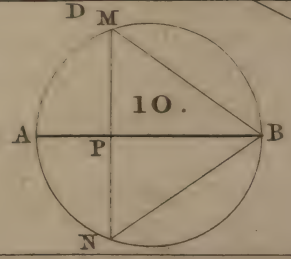
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10.
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The eaſieſt way to find the greateſt of this quantity, is to multiply the factors of the denominators together, which gives $\frac{x}{ac+cx+ax+xx}$, and dividing, by taking $cx+ax$ for the firſt term of the diviſion, and ſuppoſing $a+c=n$, gives $\frac{1}{n}-\frac{ac+xx}{nxac+cx+ax+xx}$. Now becauſe the firſt part $\frac{1}{n}$ is conſtant, and the ſecond negative; it is plain, the leſs that part is, the greater the whole product will be; but the leaſt of all is, when the numerator $ac+xx$ is equal to nothing: Whence $ac+xx=0$, gives $\sqrt{ac}=+x$, or $-x$.

Conſequently, when x is a mean proportional between a and c, that expreſſion is the greateſt poſſible; which is the ſolution of the problem mentioned in art. 471.

It is evident from what has been ſaid, that moſt of the problems of the greateſt quantities, may be ſolved by finding the greateſt rectangles, inſcribed into a determinate part of the conic-ſections; the greateſt cylinders deſcribed by rectangles which are inſcribed in theſe ſpaces may be found in the ſame manner, beſides a great many other problems of this nature; but as we intended to give only the moſt uſeful ones, and to ſhew the method by which others may be found, we ſhall leave the reſt for the curious reader to find himſelf.

SECT.

SECT. V.

Of HYDROSTATICS.

DEFINITIONS.

1. HYDROSTATICS is the ſcience which conſiders the equilibrium and preſſure of fluids.

2. A fluid is a body whoſe parts yield to any force impreſſed, and are eaſily moved one amongſt another.

3. Specific gravity is a comparative term, and ſignifies the proportion between the gravities or weights of different bodies which have the ſame magnitude.

It muſt be obſerved that the weights or gravities of bodies are always equal to the quantities of matter contained in them; that is, the weight of a cubic inch of water is to the weight of a cubic inch of gold, as the quantity of matter contained in the cubic inch of water, is to the quantity of matter contained in the cubic inch of gold.

It ſeems alſo probable, that all the primary parts of matter are the ſame in all bodies; ſince ſolids become fluid by heat, and fluids ſolid by cold.

As there are hard bodies which are elaſtic, and others which are not; ſo there are fluids which are elaſtic, and others not. A fluid is elaſtic when it may be reduced into a leſſer bulk by compreſſion, ſuch

ſuch as air; and non-elaſtic, when it cannot be reduced by any force whatſoever, ſuch as water; it is only of this latter ſort we ſhall treat of here in this ſection, and the former ſhall be conſidered in the next but one.

THEOREM I.

554. *The ſurface of a fluid contained in a veſſel will be always ſmooth and level, when it is acted upon by the force of gravity only.*

For ſince every particle is preſſed downwards by the force of its gravity; ſhould any one be higher than the reſt it would endeavour to deſcend lower, and having nothing to ſupport it, or to oppoſe its motion, will deſcend, till ſuch time it becomes even with the reſt, and remain ſo, unleſs ſome other force diſturbs it.

THEOREM II.

555. *If a ſyphon, or a tube with two branches* AFB, *be filled with an homogeneous fluid; the ſurfaces of this fluid in both branches will be in the ſame horizontal line.* Plate XXII. Fig. 1.

For if the fluid was put in motion by ſome external force, the quantity A*a*c C which paſſes through a ſection A C in one branch, muſt be equal to the quantity B*b*d D, which paſſes thro' a ſection B D in the other; and therefore the deſcent A*a* in one, will be to the aſcent B*b* in the other, reciprocally as theſe ſections A C, B D: But we have proved * that if two bodies which are in equilibrio, are put in motion, the aſcent of the one, and the deſcent of the other, are reciprocally as their weights. *art. 481.

And ſince the deſcent A*a* in the branch A C R is to the aſcent B*b*, in the branch B D S, reciprocally

cally as the section B D is to the section A C; it follows, that the fluid in one branch is in equilibrio with the fluid in the other. Consequently, the surfaces in both branches are in the same horizontal line.

Fig. 2 If one or both branches are oblique as B S, the demonstration will be the same, provided the perpendicular distance between the surfaces B D, *b d*, in the oblique branch, is taken instead of the length B *b*.

COR. I.

556. Hence, if the fluid be put in motion, the times of the ascents and descents in either branch are equal. For let A *a*, A *e*, be any two descents answering to the ascents B *b*, B *b*; then because the velocities with which the surface A C descends, and that with which the surface B D ascends, are reciprocally as these surfaces, as well as the quantities of the fluid; it follows, that the generating forces being as the velocities, the times in which these velocities are generated must be equal.

The same thing is true in regard to the waves of stagnated water.

THEOREM III.

Fig. 3. 557 *An upright vessel filled with an homogeneous fluid is pressed every where by a force proportional to the surface pressed and the height of the fluid above it.*

For let the height of the vessel A B C D be divided into any number of equal parts as C *a*, *a b*, *b* D; then it is plain, that the surface n *a* is pressed by the weight of the fluid n B C *a*, the surface m *b* pressed by the weight m B C *b*, and the bottom A D by the weight of the whole fluid.

Now

Now the preſſure of the fluid n B C *a* on the ſurface n *a* is reſiſted by the fluid below it, becauſe reaction being equal to action; it will therefore endeavour to move in that plane by a force equal to its weight, if it was not reſiſted by the ſides of the veſſel; and if there was a hole either at n or *a*, it would move through it with that force; and therefore the parts n, *a*, which prevent this motion, will ſuſtain the whole preſſure of the weight of the fluid n B C *a* above it; in the ſame manner the parts m, *b*, of the plane m *b* will be preſſed by the whole weight of the fluid m B C *b* above it. And ſince the weights of the fluids n B C *a*, m B C *b*, which have the ſame baſe, are as the altitudes n B, m C; it follows, that the fluid preſſes this veſſel every where with a force proportional to the ſurface preſſed and the height of the fluid above it.

COR. I.

558. Hence, if the line C E be drawn ſo as to meet the baſe A D produced in E; and lines *a g*, *b h*, are drawn thro' the points *a*, *b*, parallel to the baſe; then becauſe of the parallelism C *a* : C *b* :: *a g* : *b h*; it follows, that if D E expreſſes the preſſure againſt the point D; *b h*, *a g*, will expreſs the preſſure againſt the points *b* and *a*; and the area of the triangle C D E will expreſs the preſſure againſt the line C D; therefore the triangular priſm, whoſe baſe is the triangle C E D, and whoſe altitude the breadth of the ſide of the veſſel, will expreſs the preſſure againſt that ſide.

COR. II.

559. It is alſo manifeſt, that the preſſure againſt either ſide of a cubic veſſel is juſt half the preſſure

preſſure againſt the bottom: For ſince the preſſure againſt AD is as the rectangle of the ſame baſe and altitude of the triangle CDE; and that againſt DC as the triangle itſelf; it follows that the preſſure againſt the bottom is as the parallelepipedon whoſe baſe is the rectangle made by the ſides DC, DE, and altitude the breadth of the ſide, which is double the triangular priſm, whoſe baſe is the triangle CDE, and of the ſame altitude. Therefore the preſſure againſt the bottom is double the preſſure againſt the ſide.

COR. III.

560. Since the preſſure againſt the bottom is equal to the whole weight of the fluid, that againſt the ſide will be half that weight; and as there are four ſides, the preſſure againſt the four will be double the weight; and conſequently, the ſum of the preſſures againſt the ſides and bottom, is triple the weight of the fluid contained in the veſſel.

COR. IV.

561. If it was required to make the ſides of the veſſel of ſuch a ſtrength as to reſiſt every where the preſſure alike; let DE expreſs the ſtrength near the bottom, draw CE; then the lines *ag*, *bh*, will repreſent the ſtrengths at *a*, *b*.

It may be obſerved, that if a tube with a bottom fixed to it, be filled with ſo much of a fluid as to make it burſt in order to find its ſtrength, although the greateſt preſſure is near the bottom, yet it will not burſt there but ſomewhat higher; becauſe the bottom keeps the ſides together, and prevents them from giving way.

THEOREM IV.

562. *Fluids preſs every where according to their per-*

perpendicular altitudes, whatever their quantities are, or of whatſoever the figures of the veſſels in which they are contained, may be.

Let A B E F be an upright cylindric or quadrangular veſſel, with a tube of any length fixed to it: I ſay, that the bottom A F will be preſſed with a force equal to the weight of a column of the fluid, whoſe baſe is equal to that of the veſſel and altitude, to that of the veſſel and tube together. *Fig.* 4.

For ſuppoſe the fluid contained in the veſſel A B E F to be divided into columns, whoſe baſes are equal to that of the tube; then becauſe the column L n endeavours to deſcend by its weight, ſo as to become to a level with the adjacent ones, theſe by reaction will endeavour to aſcend to the ſame height, and ſo will preſs the ſurface B E with a force equal to the weight of the fluid contained in the tube C L D; theſe columns will, by the ſame reaſon, preſs the adjoining ones in the ſame manner, and they the ſurface B E. Therefore the ſurface B E is preſſed upwards by a force equal to the weight of a column of the fluid of the ſame baſe, and whoſe altitude is equal to that of the tube C L. But becauſe reaction is equal to action, the fluid cannot preſs the ſurface B E without preſſing at the ſame time the baſe A F with the ſame force; and ſince the baſe A F ſupports likewiſe the weight of the fluid A B E F, it follows, that the baſe A F is preſſed by a force equal to the weight of a column of the fluid of the ſame baſe, and whoſe altitude is equal to that of the veſſel and tube together.

This may likewiſe be proved in this manner. Suppoſe the veſſel to be continued to the height r s, of the tube; then becauſe the fluid contained

in

in the part BrsE preſſes as much downwards on the plane BE as the fluid contained in ABEF preſſes it upwards; and the baſe AF will be preſſed by the whole column ArsE: Now if the weight of the fluid on the baſe BCDE be taken away, and the reſiſtance of that plane be taken inſtead of it, it is evident, that the fluid contained in ABEF preſſes upwards againſt BE, with a force equal to that weight; but it cannot preſs upwards, without preſſing at the ſame time the baſe AF downwards with the ſame force; to which adding the weight of the fluid contained in ABEF, the baſe AF will be preſſed by a force equal to the weight of a column equal to ArsF.

This propoſition has been confirmed by experiments; for if a veſſel made ſo as to open like a pair of bellows, and a tube of any diameter be fixed to it; by pouring water into this tube, it will raiſe the upper baſe, loaded with a weight, nearly equal to that of a column of water of the ſame baſe, and an altitude equal to that of the veſſel and tube.

Fig. 5. If the veſſel is like a funnel, as ABLCD, the preſſure will ſtill be equal to the weight of a column of the fluid which has the ſame baſe and altitude; becauſe the column Ln anſwering to the tube, preſſes the bottom and the adjacent columns with its whole weight, and by reaction they preſs the bottom with the ſame force, as well as all the reſt of the columns; that is, in the ſame manner as if the veſſel were of the ſame bigneſs ArsD quite up to the top.

Fig. 6. If the veſſel is wider above than below, as ABCD, the baſe AD will only be preſſed by the column ArsD above it, and the reſt of the fluid

fluid will be ſupported by the ſides A B, D C, of the veſſel.

If the veſſel was oblique, as A *b c* D, and of any figure whatſoever, the ſame thing will be true; that is, the baſe A D will be preſſed by a column of the fluid whoſe baſe is equal to the baſe A D, and altitude to that of the veſſel. *Fig.* 7.

In general, of what figure ſoever a ſurface preſſed may be, the preſſure will always be as the column of the fluid which preſſes it, and whoſe height is the depth of the fluid.

It ſeems to be ſurprizing that a ſmall quantity of a fluid ſhould raiſe a very great weight; but when we conſider what a prodigious weight a power applied to a machine made of wheels or ſcrews may raiſe, there is no reaſon to doubt that a fluid applied in a proper manner may produce the ſame effect.

COR. I.

563. Hence, the bottom R S of the baſe of a ſyphon is every where preſſed by a column of the fluid of an equal baſe, and the ſame altitude F L of the fluid above it, and the upper part E is likewiſe preſſed upwards with a force equal to a column of the fluid, of the ſame baſe and altitude E L. *Fig.* 1.

Therefore, when pipes are made to conduct water from one place to another, regard muſt be had to the height of the head above it; otherwiſe, they may be made too weak, and break, which would be attended with unneceſſary expences.

COR. II.

564. If a rectangular ſurface A B C D, perpendicular to the horizon, be preſſed by water, ſuch as the gates of docks or ſluices; the preſſure a- *Fig.* 8.

gainſt any horizontal line ac, will be as the rectangle made by that line and the depth aA under water; and the preſſure againſt bd as the rectangle made by that line and the height bA. Therefore the preſſure againſt the whole ſurface will be equal to a triangular priſm of water whoſe baſe is equal to the ſurface ABCD, and altitude half the depth AD of the water. Conſequently, this preſſure is expreſſed by the ſolid $\frac{1}{2}$ AB × $\overline{AD}^2$.

If AB = 20 feet, and AD = 12, that ſolid will contain 1440 cubic feet: Now as a cubic foot of common water weighs 62.5 pounds, it is evident that the plane is preſſed by a weight of 1440 × 62.5, or 90000 pounds.

COR. III.

Fig. 9. 565. If the ſurface be oblique, ſuch as is repreſented by ABCD, An being the ſurface of the water; by drawing 1a, mb, nB, perpendicular to An, and ac, bd, Be, perpendicular to AB, each equal to the reſpective depth; then it is evident that the priſm of water, whoſe baſe is the triangle ABe, and altitude the breadth of the plane ABCD, will expreſs the preſſure of the water againſt that plane; and any part of that priſm, anſwering to the baſe caBe, that againſt the plane aBCq.

PROB. I.

Fig. 10. 566. *Let* AD *be the breadth of a river or canal; to find the poſition of the gates* AE, ED, *ſuch as to reſiſt the preſſure of the water with the greateſt force poſſible.*

On AD as a diameter, deſcribe a ſemi-circumference of a circle, draw the radius CB perpendicular to AD; produce DE ſo as to meet the circum-

circumference in M, and join A M. Now as the preſſure againſt D E or A E is as that line D E; and as it becomes greater, its force decreaſes and the preſſure increaſes; it is evident that the ſtrength of the gate is reciprocally as the line E D; but by the ſimilarity of triangles, we have D E : D C :: D A : D M; and ſince D C and D A are conſtant, D M will expreſs the ſtrength of the gate A E or E D; and becauſe the ſtrength of timber of the ſame diameter decreaſes, as it increaſes in length; the ſtrength of the gates, in reſpect to that of the timber and the preſſure together, will be expreſſed by $\overline{DM}^2$.

Again, the force with which the gate A E reſiſts the preſſure of D E at E, is equivalent to the forces E M and A M, the former parallel, and the latter perpendicular to D E. Therefore, the force with which one gate reſiſts the preſſure of another, is as A M. Conſequently, $AM \times \overline{DM}^2$ ought to be the greateſt poſſible.

Hence, if $AD = a$, $AM = x$; then will $\overline{DM}^2 = aa - xx$, and $AM \times \overline{DM}^2 = aax - x^3$ which will be the greateſt of all, when $3xx = aa$, by art. 546; for if the line 2 A D (2 a) be divided equally and inequally, ſo as the product of the rectangle $a + x$ by $a - x$, be multiplied by the middle part x, be the greateſt poſſible, we ſhall find the value above of x. Conſequently, A D (a) is to A M ($a\sqrt{\frac{1}{3}}$) as the radius 100000 is to the ſine 57735 of the angle A D M, which will be found to be 35 degrees and 16 minutes.

THEOREM V.

567. *The quantity of preſſure againſt any ſur-* Fig. 8.

 face

face is equal to the solid whose base is the surface, and altitude the distance of its center of gravity, from the surface of the fluid.

I. Let the rectangular surface A B C D be perpendicular to the horizon, its center of gravity
*art 422. will be * in the middle of the line which bisects the sides A D and B C: Therefore $\frac{1}{2}$ A D will be its distance from the surface, and $\frac{1}{2}$ A B × $\overline{\text{A D}}^2$;
*art. 564. the quantity of the pressure, which is the * same as before.

Fig. 9. II. Let the surface be oblique as A C; its center of gravity will be in the perpendicular which bisects A B; and hence its distance from the surface A n of the fluid, will be equal to half of B n or B *e*; and calling D the breadth of the surface,
*art. 564. we shall have * $\frac{1}{2}$ B *e* × A B × D for the pressure.

III. Let *a* B be the side of an oblique surface, whose breadth is D, its center of gravity will be in the perpendicular which bisects *a* B; let the point *b* bisect *a* B; then *b* m, or its equal *b d*, will be the distance of this center from the surface, and
art. 564. * $\frac{1}{2}$ *b d* × *a* B × D, the quantity of pressure against that surface.

THEOREM VI.

568. *The quantity of pressure on any solid is equal to the sum of all the surfaces multiplied by the distance of their common center of gravity from the surface of the fluid.*

Let any number of weights be suspended by
Fig 11. their centers of gravity *a*, *b*, *c*, *d*, to any horizontal plane A B, by lines *a o*, *b o*, *c o*, *d o*; draw a line from *a* to *b*, in which take the point *x* so as *a* : *b* : : *b x* : *a x*; then will *x* be the common center of gravity of the weights *a* and *b*; and if to

the perpendicular ox, the lines am, bn, are drawn perpendicular; the similar triangles amx, bnx, give $ax : bx :: mx : nx :: b : a$, by supposition; therefore $mx \times a = nx \times b$; and because $ao = mo$, $bo = no$, and $mx = mo - xo$, $nx = xo - no$; we have $a \times \overline{mo - xo} = b \times \overline{xo - no}$, by equal substitution; or $a \times ao + b \times bo = \overline{a + b} \times xo$, by transposition: Therefore the sum of the products of the two weights a, b, multiplied by their distances from the surface, is equal to the sum of these weights multiplied by the distance of their common center of gravity x, from that surface.

Join the weight c to the center of gravity x, and let the point y be such as $a + b : c :: cy : xy$; then will y be the common center of gravity of the weights $a + b$ and c. Therefore, by what has been said, $\overline{a + b + c} \times yo$, will express the sum of the products of the weights a, b, c, multiplied by their respective distances from the plane A B.

Lastly, join the point d to the point y, and take the point z, such that $a + b + c : d :: dz : yz$; then will the point z be the common center of gravity of the weights $a + b + c$, and d; and by what has been proved $\overline{a + b + c + d} \times zo$ will express the sum of all the weights multiplied by their distances from the plane A B.

Hence, if a surface or number of surfaces of any kind be considered as equally ponderous in every equal part, and as divided into indefinitely small parts, suspended by lines, drawn from their centers perpendicular to any horizontal plane; it is manifest, that if every part be multiplied by its distance from that plane, the sum of the products will be equal to the product of the whole surface or surfaces, multiplied by the distance of their

common center of gravity from the laid plane; and that this equality will ſubſiſt, even if the ſaid lines be perpendicular to any plane, though not parallel to the horizon.

COR. I.

569. Hence, becauſe the center of gravity of a ſphere, right cylinder, priſm, or cube, is the ſame as the common center of gravity of their ſurfaces; the preſſures upon theſe ſolids immerſed in a fluid, will be equal to the product of their ſurfaces, multiplied by the diſtance of their center of gravity from the ſurface of the fluid.

COR. II.

570. It is alſo manifeſt, that if the center of gravity of any body immerſed in a fluid, be in the ſame vertical line with the common center of gravity of its ſurface, that ſolid will either remain at reſt, or deſcend in that vertical line; and it will never be at reſt, unleſs theſe centers are in the ſame vertical line.

Of BODIES *immerſed in* FLUIDS, *and their* SPECIFIC GRAVITIES.

THEOREM VII.

571. *The weights or gravities of bodies are in the compound ratio of their magnitudes and denſities.*

For the denſer bodies are the more matter they contain, and the greater their magnitudes are, the more matter they alſo contain. Therefore, as the weights or gravities of bodies are equal to the quantities of matter they contain, the weights of bodies

bodies are in the compound ratios of their magnitudes and denſities.

COR. I.

572. Hence it follows, that if two bodies are equal in magnitude, their weights are as their denſities; and if their denſities are equal, the weights are as their magnitudes: But if their weights are equal, their denſities are reciprocally as their magnitudes.

COR II.

573. Since the ſpecific gravity of bodies, are as the weights of equal magnitudes, they are therefore as their denſities. Conſequently, whatever is ſaid in reſpect to the denſities of bodies, may alſo be underſtood in regard to their ſpecific gravities.

THEOREM VIII.

574. *If a body be immerſed in a fluid, it will remain in equilibrio, ſwim, or ſink, according as its ſpecific gravity is equal, leſs, or greater, than that of the fluid.*

For a body immerſed in a fluid will preſs downwards by the force of gravity; but as it cannot deſcend without moving as much of the fluid out of its place as is equal to it in bulk; it is evident, that it will be preſſed upwards by the weight of as much of the fluid as it removes from its place; and therefore, if the ſpecific gravity of the body is equal to that of the fluid, it will remain at reſt in any depth; ſince the fluid cannot be compreſſed by ſuppoſition, and the force of gravity is the ſame, whether the ſame ſpace be filled by the fluid or by the body.

If the ſpecific gravity of the body be greater than that of the fluid, it muſt of conſequence deſcend, by the exceſs of its weight above that of the fluid; and if it be lighter, it will be preſſed upwards by a greater force than that whereby it endeavours to deſcend; and therefore it will aſcend with a force equal to the difference between the ſpecific gravities of the fluid, and its own.

COR. I.

575. Hence, if the ſpecific gravity of a body immerſed, be leſs than that of the fluid; it will ſink ſo much only as to fill up a ſpace, which contains as much of the fluid as is equal in weight to it. For if the body weighs juſt as much as the fluid which takes up the ſame ſpace, all the ambient parts of the fluid will be preſſed in the ſame manner as if that ſpace was filled with the fluid, and conſequently there will be an equilibrio between the fluid and the body.

COR. II.

576. Whence, the magnitude of the whole body is to the magnitude of the part immerſed, as the ſpecific gravity of the fluid is to that of the body; ſince when the weights of two bodies are equal, their denſities, or which is the ſame, their *art. 573. * ſpecific gravities, are reciprocally as * their *art. 572. magnitudes.

COR. III.

Fig. 12. 577. If a body *b* ſpecifically heavier than the fluid be immerſed and ſupported by a power P, this power will be equal to the exceſs of the ſpecific gravity of the body above that of the fluid; ſince

ſince the body would deſcend by a force * equal to that exceſs. *art. 574.

COR. IV.

578. Conſequently, if the weight of a body immerſed in a fluid be ſubtracted from its weight in the air, the difference will be the weight of an equal bulk of the fluid, and therefore is to the weight in the air, as the ſpecific gravity of the fluid is to that of the body.

Examp. Let a piece of copper weigh 9 ounces in the air, and 8 in water; then the difference of one is to 9 as the ſpecific gravity of water is to that of copper.

COR. V.

579. Hence, if a expreſſes the ſpecific gravity of the body A, whoſe weight in water is B; and c the ſpecific gravity of water; then by the laſt corollary we have $A - B : A :: c : a = \frac{cA}{A-B}$. Conſequently, the ſpecific gravities of bodies will be as their weights in the air directly, and their loſs in the ſame fluid inverſely; ſince the ſpecific gravity c of the fluid is conſtant, and $A-B$ expreſſes the loſs of the weight in the fluid.

COR. VI.

580. If the ſame body be weighed in ſeveral fluids, and is ſpecifically heavier; then becauſe $a = \frac{cA}{A-B}$, gives $\frac{a \times A-B}{A} = c$, or only $A - B = c$; ſince a, A, are here conſtant: It is evident, that the ſpecific gravities of theſe fluids are to one another as the weights of the body loſt in the fluids.

RE-

REMARK.

From what has been ſaid in regard to the ſpecific gravities of bodies, ſolid or fluid, in the three laſt corollaries; the ſpecific gravities of bodies may eaſily be found by means of an hydroſtatic ballance; but as they have been found already by curious perſons, eſpecially by Mr. *Cotes*, in his hydroſtatic lectures, we ſhall inſert a part of his tables, which, in my opinion, are the beſt of this kind, leaving to thoſe who have an opportunity the pleaſure of trying them over again.

A TABLE *of* Specific Gravity *of* Bodies.

Fine gold	19640	Dry boxwood	1030
Standard gold	18888	Sea water	1030
Quic-ſilver	14000	Common water	1000
Lead	11325	Dry oak	0925
Fine ſilver	11091	Dry aſh	0800
Standard ſilver	10535	Dry maple	0755
Copper	9000	Dry elm	0600
Caſt braſs	8000	Dry fir	0550
Gun metal	8784	Cork	0240
Steel	7850	Air	0001 $\frac{1}{4}$
Iron	7645	Sand	1520
Caſt iron	7425	Light earth	1984
Tin	7320	Loom	2160
Stone of mean gravity	2500	Marble	2700
Brick	2000	Pitch	1150

N. B. We have taken ſo much out of Mr. *Cotes*'s table as we imagined might be moſt uſeful to an engineer, and have added the ſpecific gravities of gun metal, caſt iron, and of the different kinds of earth. It has been found by experiments,

ments, that a cubic foot of common water weighs 1000 ounces averdupois; and therefore, the numbers in the table expreſs not only the ſpecific gravities of the bodies mentioned therein, but likewiſe the exact weights of a cubic foot of each; and by proportion the weight of a greater or leſs bulk of theſe bodies may be found.

This table ſerves likewiſe to find the magnitudes of irregular bodies; for inſtance, ſuppoſe a piece of dry elm weighs 20 pounds, or 320 ounces, ſay as 600 ounces is to 320 ounces, ſo is a cubic foot or 1728 cubic inches to the magnitude required, which is 921.6 cubic inches.

PROB. VI.

581. *The ſpecific gravities of two ingredients of which a mixture is made being given, as likewiſe the ſpecific gravity of the mixture, to find the quantity of each ingredient contained in that mixture.*

Let a, b, expreſs the ſpecific gravities of the ingredients, c the weight of the compound, d its ſpecific gravity; then if z expreſſes the weight of the ingredient whoſe ſpecific gravity is a, $c-z$ will be that whoſe ſpecific gravity is b; and $\frac{z}{a}$ $\frac{c-z}{b}$, will be * the weights loſt in the fluid, and $\frac{c}{d}$ that of the mixture, which muſt be equal to the ſum of the former. Hence, $\frac{z}{a}+\frac{c-z}{b}=\frac{c}{d}$; and therefore $z=\frac{ac}{d}\times\frac{b-d}{b-a}$.

* art. 579.

Examp. Let a mixture of copper and tin, weigh 112 pounds, whoſe ſpecific gravity is found

found to be 8784, that of tin 7320, and that of copper 9000; then will $a=7320$, $b=9000$, $d=8784$, $c=112$, and $b-d=216$, $d-a=1464$, and $b-a=1680$. Therefore $z=\frac{7320\times 112\times 216}{8784\times 1680}=12$, will be the number of pounds of tin in the mixture, and $112-12=100$, the number of pounds of copper.

COR. I.

582. It is also manifest, that having the specific gravities of the parts, that of the mixture may be found; for because $z=\frac{ac}{d}\times\frac{b-d}{b-a}$, we get $bdz-adz=abc-acd$, by multiplication; and $d=\frac{abc}{bz-az+ac}$. Consequently, having the weights of the parts and their specific gravity, the weight of the compound being given, its specific gravity is given likewise.

COR. II.

583. If the weights of the ingredients are called m, n; then will $m=z$, $n=c-z$; and the equation $\frac{z}{a}+\frac{c-z}{b}=\frac{c}{d}$, gives $\frac{m}{a}+\frac{n}{b}=\frac{c}{d}$. Hence we get $a=\frac{bdm}{bc-dn}$: or, $\frac{c}{d}-\frac{n}{b}:m::1:a$.

This equation furnishes a means to find the specific gravity of bodies, specifically lighter than the fluid: For if to this body another is joined much heavier, whose specific gravity is known; then finding the weight of the compound, as likewise its specific gravity, and dividing the weight

m, of the lighter body by the * difference $\frac{c}{d} - \frac{n}{b}$; * art. 572.
between the magnitudes of the compound and the heavier body, the quotient will give the ſpecific gravity of the lighter body.

Examp. If a piece of elm weighs 15 grains, having joined to it a piece of copper of 18 grains; let the ſpecific gravity of the compound be $\frac{11}{9}$; then will $m = 15$; * $\frac{c}{d} = 27$; and $\frac{n}{b} = 2$. * art. 583.

Whence $\frac{c}{d} - \frac{n}{b} = 25$, and $a = \frac{15}{25} = 0.6$ equal to the ſpecific gravity of elm; ſuppoſing that of copper to be 9.

SECT. VI.

Of HYDROLICS.

HYDROLIC is the ſcience which conſiders the motion of fluids, and the forces with which they act upon machines.

The uſefulneſs of this ſubject has induced ſeveral eminent men to write on it; the moſt noted among the moderns, are Sir *Iſaac Newton*, Mr. *Mariotte*, Dr. *Jurin*, *Daniel Bernoullie*, and Mr. *Belidor*; but as it is the moſt difficult branch of the mathematics, it ſo happens, that the true eſtimate of the quantity of water, diſcharged thro' an orifice in any given time, has not as yet been rightly determined. What Sir *Iſaac Newton* has ſaid

* *Prop 36. book 2. princip.* said on this subject * extends to the case only, when the orifice is very small; Mr. *Mariotte* gives some very curious experiments; but his rules are far from being true: Dr. *Jurin* has published of late something on this subject, which is not as yet come to my hands; in his former book * he chiefly endeavours to prove Sir *Isaac Newton*'s rule: And as to what Mr. *Belidor* has done on this account in his *Architecture Hydrolics*, is not sufficiently well grounded to be depended upon; he has even made some visible mistakes, as we shall shew hereafter.

* *Dissert. physic. mathem.*

Daniel Bernoullie has taken great pains in his book called *Hydrodinamicæ*, to determine all the different cases of running water; but by a mistaken notion of some new principle, he has not better succeeded than those who have treated this subject before him, as will be made appear hereafter.

Since, therefore, the endeavours of so many great men have not been quite so successful as might have been expected from their extensive knowledge in the mathematics; we dare not presume to think that we have entirely freed this subject from all its difficulties; but hope that the reader will look upon what we shall say on this account, rather as an essay on this important subject, than as a compleat treatise.

THEOREM I.

Fig. 13. 584. *The velocities of the water in any two vertical sections* AB, CD, *of a river or canal* EF, *are reciprocally as the areas of these sections.*

For the quantity of water running through one in any given time, is constantly equal to that which runs through the other in the same time, otherwise

otherwiſe the water would riſe in one place and fall in another, which is contrary to experience: And becauſe the quantity of water which runs thro' a ſection in a given time, is equal to a column whoſe baſe is equal to the ſection, and altitude to the ſpace run over by a body in the ſame time with the velocity of the water: It follows, that the quantity of water run through a ſection is in the compound ratio of the ſection and velocity; therefore, ſince the quantities of water run through any two ſections, at the ſame time, are equal, and the altitudes of equal ſolids * reciprocally proportional to their baſes, and theſe altitudes being as the velocities, the velocities are reciprocally as the ſections.

* art. 204.

COR.

585. If a expreſſes the height through which a falling body may acquire the velocity of the water running through the ſection A B, and b that thro' which a falling body may acquire the velocity of the water in the ſection C D; then becauſe the velocities of falling bodies are * as the ſquares of the heights through which they fall, we have $a : b :: \overline{C D}^2 : \overline{A B}^2$. Conſequently, the heights through which falling bodies may acquire the velocities of running water, are reciprocally as the ſquares of the ſections.

* art. 450.

REMARK.

If the velocity in every part of the ſame ſection was the ſame, it would be an eaſy matter to find the quantity of water diſcharged by a river or canal in any given time; but experience ſhews the contrary; for at the ſides, or near the bottom, it will run much ſlower than in the middle of the ſtream,

ſtream, as being continually retarded by friction; in order therefore to find the true velocity by which the diſcharge is to be eſtimated; that at the ſides, middle, and near the bottom, muſt be found, then the mean between them will be the velocity required.

If for example, the velocity in the middle of a ſtream is ſuch as to carry a body over a ſpace of 12 feet in a minute, near the bottom 8, and 6 at the ſides; then the ſum 26 of theſe three numbers, divided by 3, gives $8\frac{2}{3}$ feet run through by mean velocity in a minute.

The beſt way of finding the velocity of a ſtream is, to let an empty bottle, cork'd, ſwim in it; and to make it go near the bottom, ſome water muſt be let into it, ſo as to be very little ſpecifically heavier than water, and then to tie a piece of cork to it, which will keep it from ſinking, and at the ſame time ſhew the velocity of the water.

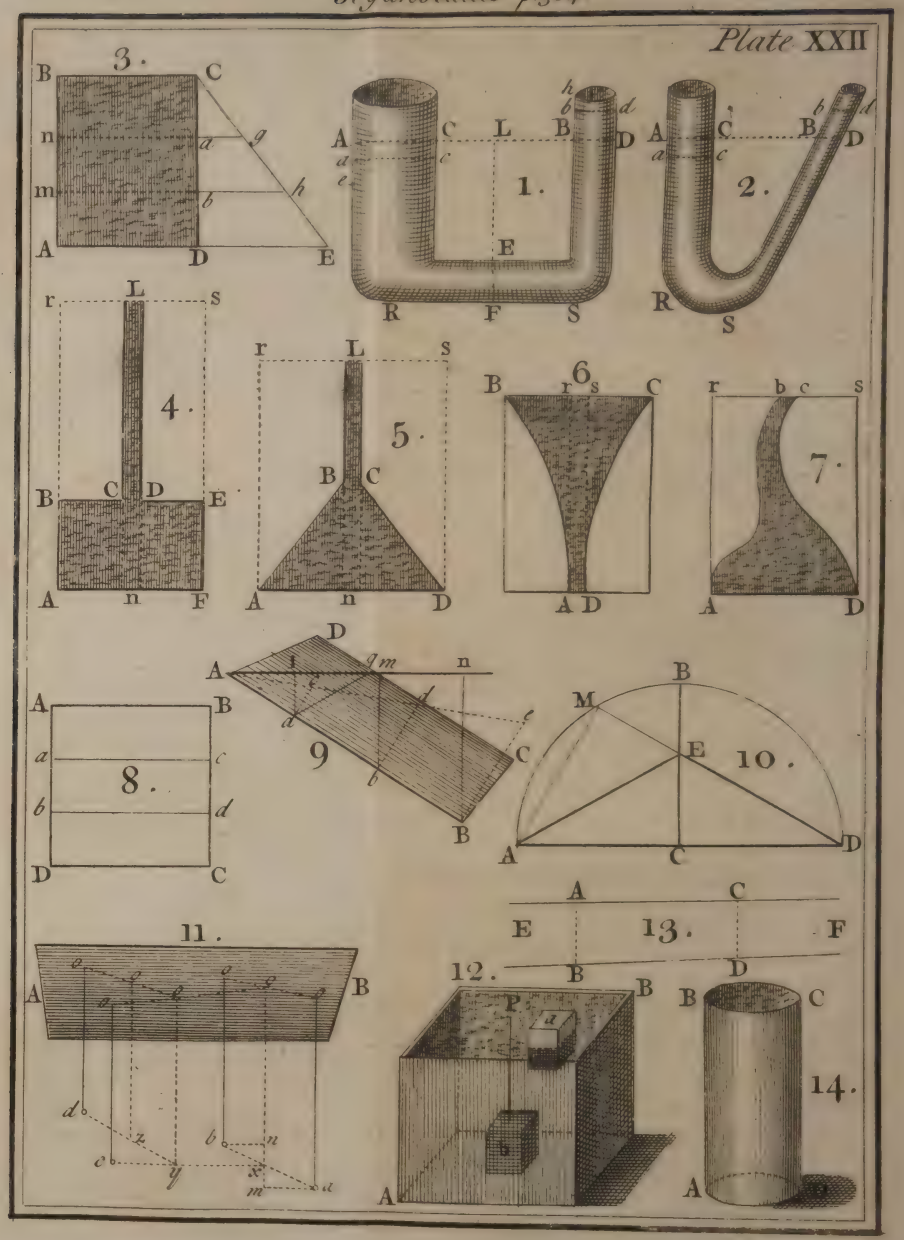
THEOREM II.

Fig. 14. 586. *If a right cylinder* A B C D *be conſtantly kept full, the velocity with which the water runs out at the bottom, will be equal to that which a falling body would acquire by deſcending through the height of the cylinder.*

For ſince water deſcends by the force of gravity in the ſame manner as all other bodies, and there is no other obſtacle to retard its motion by ſuppoſition; it is evident that the velocity with which the water runs out at the bottom, is equal to that which a body would acquire by falling through the height A B of the cylinder.

It may be obſerved that the cylinder muſt not be too ſmall, otherwiſe the adheſion of the water to the ſides will retard its motion, as appears in capillary

Plate XXII



capillary tubes; beſides, this rule takes place only after the upper ſurface BC is deſcended to the bottom.

COR. I.

587. It is manifeſt, that the velocity of water at the bottom is conſtant and uniform; for whatever the velocity of any ſection, when deſcended to the bottom, may be; that of any other which deſcends through the ſame height, muſt be the ſame.

COR. II.

588. If the cylinder be oblique, ſuch as EFGH, the velocity of the water at the bottom, will be the ſame as that which runs through a right cylinder of the ſame baſe EH and altitude FL; for we have proved * that the baſe EH is preſſed by the ſame weight as if the cylinder was upright, and therefore the generating force being the ſame, the velocities produced muſt be ſo too. *Plate* XXIII. *Fig.* 15.

* *art.* 562.

COR. III.

589. The velocity of the water at the baſe, when the cylinder is conſtantly kept full, is double the velocity when no freſh ſupply of water is let in above. For we have proved, that the water runs out at the bottom with an uniform velocity equal to that acquired by a body falling through the height of the cylinder, when the water was kept at the ſame height; and it is plain, that when the water deſcends through the cylinder without being ſupplied with any other water, the velocity at the bottom will increaſe in the ſame manner as a body falling from a ſtate of reſt, and therefore will run through half the ſpace in the

same time that a body would describe with an uniform velocity equal to the velocity acquired during the fall.

PROB. I.

Fig. 16. 590. *To find the velocity with which water runs out of an orifice* LM *made in the bottom of an upright vessel* ABCD.

Case I. If the vessel be constantly kept full, in AB, take the point K such as twice the square of the surface EF, minus the square of the orifice is to the square of the surface EF as the height AB is to AK; that is, if EF and LM express the areas, and $2\overline{EF}^2 - \overline{LM}^2 : \overline{EF}^2 :: AB : AK$; then the fourth term AK will be the height thro' which a falling body will acquire the velocity with which the water runs out at the orifice.

Case II. If the vessel is not supplied with water whilst that which is in it runs out; then if $3\overline{EF}^2 - \overline{LM}^2 : \overline{EF}^2 :: AB : AK$; the height AK will be such as a body falling through it, will acquire the velocity with which the water runs out at the orifice.

As the demonstration of this problem depends on the method of fluxions, we shall first draw all the consequences which follow naturally from it; and for the sake of those who understand that method, give the demonstration afterwards.

COR. I.

591. Hence, if the orifice is but small in comparison to the surface EF of the vessel, then $2\overline{EF}^2$ may be taken for $2\overline{EF}^2 - \overline{LM}^2$, in the first case, without any sensible error in practice; and

and therefore $2\overline{EF}^2 : \overline{EF}^2 :: AB : AK = \frac{1}{2}AB$, which agrees with Sir *Isaac Newton*'s rule †. And when the orifice is equal to the surface EF; that is, when the vessel has no bottom, then the proportion in the first case becomes $\overline{EF}^2 : \overline{EF}^2 :: AB : AK = AB$, which agrees with what has been demostrated in art. 586.

And in the second case the proportion becomes $2\overline{EF}^2 : \overline{EF}^2 :: AB : AK = \frac{1}{2}AB$, which agrees with what has been said in art. 589.

COR. II.

592. It is therefore manifest, that the velocity of the water running out of an orifice in the bottom of a vessel kept constantly full, can never be less than that acquired by a falling body through half the height of the water above the orifice, nor greater than that acquired by falling through the whole height.

COR. III.

593. If x expresses the height through which a body falls from a state of rest, in a second of time, and $AK = h$; then the velocities acquired by a body falling through the heights x, h, will be * as their square roots. And because a body describes a space double, * uniformly with the velocity acquired in the fall at the same time; it is evident that the velocity $\sqrt{x}$ is to the velocity $\sqrt{h}$, as the space $2x$ described uniformly with the velocity acquired in a second, is to the space $\frac{2x\sqrt{h}}{\sqrt{x}}$ or $2\sqrt{xh}$ described uniformly in the same time with

* art. 450.

* art. 451.

† *Principia prop.* 36, *book* II.

the velocity acquired by falling thro' the height AK.

COR. IV.

594. Since the quantity of water discharged through an orifice in any given time, is in the compound ratio of the velocity and the surface of the orifice; if we call ⊙ the orifice LM; then will $2 \odot \sqrt{x}\,h$ express the quantity of water discharged in a second.

COR. V.

595. Because a body falls through a space of 16.1 feet, or 193.2 inches, from a state of rest, in a second, in the latitude of *London*; if we suppose $x = 193.2$, we shall have $2 \odot \sqrt{193.2}\,h$, for the quantity of water discharged in a second, expressed in inches, supposing the height AK, and the orifice LM, to be expressed in inches.

COR. VI.

596. Since a body falls through a space of 15 feet, or 180 inches, *French* measure, in a second, in the latitude of *Paris*, as has been found by experiments, if we suppose $x = 180$, then will $2 \odot \sqrt{180}\,h$ express the quantity of water discharged in a second expressed in *French* measure.

EXAMPLE I.

597. Let the height of the water in a reservoir, above the orifice, be 13 feet *French* measure, and the diameter of the orifice .25 parts of an inch; then because the orifice is but small, in comparison to the breadth of the vessel, we may suppose * $h = 6.5$ feet, or $h = 78$ inches. Therefore

*art. 591.

fore $180h = 14040$, or $\sqrt{180h} = 118.4$; and since the diameter of the orifice is .25 inches, its area $\odot$ will be $= .049$. Hence, if these values are substituted into * $2 \odot \sqrt{180h}$ we shall have 11.6 cubic inches of water discharged in a second, or $11.6 \times 60 = 696$ inches in a minute. *art. 596.

Mr. *Mariotte* has found § by several experiments, that a reservoir of such dimensions as we have supposed here, discharged 14 pints of water in a minute, or because he supposes 35 pints in a cubic foot, 691.2 cubic inches; which comes very near to what we have found by the theory. The experiments will always give less than the true quantity, if the water is not let to run first, till it has acquired its due velocity before it is collected.

EXAMPLE II.

598. Let the height of the water in a reservoir be 25 inches *English* measure, and the orifice a square inch; by supposing $h = 12.5$, and $\odot = 1$; we shall have * $2 \odot \sqrt{193.2h}, = 98$ cubic inches nearly. *art. 595.

Dr. *Desagulier* has found * by repeated expements, as he says, that a vessel filled with water to the height of 25 inches, discharged 5.2 tuns in an hour, through a square orifice of an inch. Now because a tun of wine measure contains 58212 cubic inches, 5.2 makes 302702.4 cubic inches, which being divided by 3600, the number of seconds in an hour, gives about 84 cubic

§ *Regles des jets d' eau*, p. 486.
* *Page* 128, *vol.* II.

inches of water discharged in a second, which is less by 14 inches than what the theory gives.

As the Dr. has not given the breadth of the vessel, nor mentioned what measure he used, it is hard to know from whence this great difference arises. But as our theory agrees with Mr. *Mariotte*'s experiments, which are known to have been made with great care and exactness, I am satisfied that the rule we have laid down will always answer nearly such experiments as are made with accuracy.

If the orifice be made in the side of the vessel, the rule laid down in the last problem will hold good in this case likewise, provided the mean velocity be taken; for the water will run faster near the bottom than above, when the orifice is but small, the height of the fall may be taken from the center; but when it is large, the mean velocity must be found in the manner that will be shewn hereafter.

If the vessel be oblique, and its width taken horizontally every where the same, the foregoing rule will still hold good. For it has been proved * that the bottom of such vessels, or any horizontal section, is pressed in the same manner as an upright one of the same base and altitude; and therefore the generating force being the same, the velocities generated must be equal.

* *art.* 562.

Demonstration *of the foregoing* Problem.

Case I. As the particles of water, or those of any other fluid whatsoever, have different velocities in different sections, it is plain that the quantity of water discharged through an orifice in any given

given time must be estimated by a mean velocity, and not by that of any particular part. And because it has * been proved that the distance of the center of gravity of a body from any plane, multiplied by the quantity of matter of that body, is equal to the sum of the products of all the particles each multiplied by its respective distance from that plane, it is manifest, that the descent of the center of gravity of the water contained in the vessel AEFD, during the discharge of that quantity through the orifice, is equal to that through which a falling body will acquire the mean velocity of the water.

* art. 568.

Whence, if v expresses the descent of the center of gravity, while the surface EF descends from E to A, and z the quantity of water in the vessel; then will zv expres the momentum of that quantity of water, whose fluxion is $\dot{z}v + z\dot{v}$.

Now if we suppose that while the surface EF descends through an indefinite small space E*e* so as to become in the situation *ef*, the quantity of water passed through the orifice LM, to be expressed by the column OLMN; then by adding LMNO to the quantity AEFB, and subtracting EF*fe*, we shall have AEFD + LMNO — EF*fe*, equal to the quantity OLA*ef*DMN; and as the quantity AEFD is constant, the fluxion of the quantity OLA*ef*DMN, will be equal to the fluxion of the difference LMNO — EF*fe*.

Hence, if the surface of the orifice LM be n, that of EF m, and the descent EA of the surface EF, x; then because x expresses the descent of EF at LM, we have * $mm : nn :: x : \frac{nnx}{mm}$, equal

* art. 584

to the descent thro' ef; and therefore $x - \frac{nnx}{mm}$, expresses the difference between these two descents, whose fluxion is $\dot{x} - \frac{nn\dot{x}}{mm}$; but because mx expresses the quantity of water contained in the vessel AEFD, we have $mx = z$, and $\dot{z} = m \times \dot{x} - \frac{nn\dot{x}}{mm}$ these quantities being substituted into $\dot{z}v + z\dot{v}$ give $mv\dot{x} - \frac{nnv\dot{x}}{m} + mx\dot{v}$ for the fluxion of the momentum of the quantity of water contained in OLAefDMN. But $m\dot{x}$ is the fluxion of the quantity of water AEFD, and this fluxion multiplied by the distance x, or descent EA of the surface EF, gives $m\dot{x}x$ for the fluxion of the momentum likewise *. Therefore these two fluxions must be equal: Whence $mv\dot{x} - \frac{nnv\dot{x}}{m} + mx\dot{v} = m\dot{x}x$; or, if $1 - \frac{nn}{mm} = r$, and we divide by m, that fluxion becomes $r\dot{x}v + x\dot{v} = x\dot{x}$; and if we multiply both sides by x^{r-1}, we shall have $rv\dot{x}x^{r-1} + \dot{v}x^r = \dot{x}x^r$; whose fluent is $vx^r = \frac{x^{r+1}}{1+r}$; or dividing by x^r, $v = \frac{x}{1+r}$; but $1 - \frac{nn}{mm} = r$, or $2 - \frac{nn}{mm} = 1 + r$. Therefore

* *art.* 480.

$v =$

$v = \frac{mmx}{2mm - nn}$. Consequently the rule is true.

Case II. We have proved that $\dot{x} - \frac{nn\dot{x}}{mm}$, expressed the fluxion of the descent of the water AEFD, into the situation OLA*ef*DMN; this is true upon supposition that the water is constantly kept to the same height in the vessel; but when it descends without being supplied with fresh water, that fluxion will be $2\dot{x} - \frac{nn\dot{x}}{mm}$. For in the first case the velocity of the section of water EF, in any situation *ef* is always constant, and then we have * $mm : nn :: x$, to the height of the descent through E*e*; but when that surface EF descends only by its own gravity, it is manifest that the velocity acquired in the descent through E*e* is accelerated, whilst that in the descent thro' LO is uniform during a small space of time; and therefore describes twice the space in the same time, that it would describe by an accelerated motion; so that to find the true difference, we must subtract the descent through E*e*, from twice the descent through LO, in order to have $2\dot{x} - \frac{nn\dot{x}}{mm}$ for the fluxion in the second case, which gives $2 - \frac{nn}{mm} = r$, and $rv\dot{x}x^{r-1} + vx^r = \dot{x}x^r$ whose fluent is $vx^r = \frac{x^{r+1}}{1+r}$, as before; and by division and substitution we get $v = \frac{mmx}{3mm - nn}$ which proves the truth of the second rule.

* *art.* 585.

We have proved by the general rule, and by a par-

particular propoſition, that if the orifice LM be equal to the ſurface EF, that is, when the veſſel has no bottom, the height of the deſcent v, is equal to the height of the water, when the veſſel is conſtantly kept full, or to half that height only when it is not kept full; and this may alſo be proved from the fluxionary expreſſions: For if we ſuppoſe $m=n$, then will $1-\frac{nn}{mm}=r=0$, in the firſt caſe; and therefore the fluxion $r\dot{x}v+x\dot{v}=x\dot{x}$, becomes $x\dot{v}=x\dot{x}$, or $\dot{v}=\dot{x}$, whoſe fluent is $v=x$; and if $m=n$, then $2-\frac{nn}{mm}=r=1$, in the ſecond, whence the fluxion $rv\dot{x}+vx=x\dot{x}$, becomes $v\dot{x}+x\dot{v}=x\dot{x}$, whoſe fluent is $vx=\frac{1}{2}xx$, or $v=\frac{1}{2}x$.

Since therefore, our general rules are true in all caſes that poſſibly can happen, and perfectly agree with what has been proved in a particular manner, as well as with the beſt experiments we could get, there is not the leaſt reaſon to doubt, but that it will anſwer the practice in every particular, provided they are rightly applied.

Daniel Bernoullie has found † by ſuppoſing $r=1-\frac{mm}{nn}$, that $rv\dot{x}+x\dot{v}=-x\dot{x}$, expreſſed the fluxion containing the relation between the deſcent v of the center of gravity, and that of the deſcent x of the ſurface EF, the fluent of which is $v=\frac{mma}{2nn-mm}\times$ by $\frac{a}{x}\Big|^{r}-\frac{x}{a}$. Suppoſing a

† *Page* 37, *Hydrodynamica*.

to

to expreſs the firſt height of the water above the orifice, when the veſſel was not kept full; and when $m = n$, this expreſſion becomes $v = a - x$, which is the only caſe in which it is true, and that when the water is kept at the ſame height in the veſſel. When $2nn$ is leſs than mm, it is plain that his expreſſion becomes $v = \frac{mma}{mm - 2nn} \times$ by $\frac{x}{a}$ $- \frac{a}{x}\Big|^{r}$; which ſhews that x muſt be greater than the firſt height of the water a, which is impoſſible.

There are ſeveral miſtakes in the fluxion $rv\dot{x} + x\dot{v} = -\dot{x}x$: Firſt, he takes the deſcent of the center of gravity in a different direction from that of the parts, which is not ſo, for this rule is only true, as we have proved, when theſe diſtances * are all taken from the ſame plane. In the next, he takes the fluxion of the deſcent thro' LO to be $\frac{mm}{nn}\dot{x}$, whereas we have proved it to be $\dot{x}$ only; then he ſubtracts this from that of the deſcent through Ee, which being leſs than the former, makes the difference $\dot{x} - \frac{mm}{nn}\dot{x}$ negative; beſides, he did not conſider that when the veſſel is not kept conſtantly full, that the deſcent thro' Ee is accelerated, whilſt that through LO is uniform, during a ſmall ſpace of time, whereby he concludes quite the contrary from what he ought to do, that this expreſſion is for the deſcent, when the veſſel is kept full; and as to the caſe when the veſſel is kept full, he gives another ſolution, * which, if I am not miſtaken, is entirely groundleſs from one end to the other.

* art. 568.

* page 93.

The

The expreſſions given for the velocity of running water, both by Sir *Iſaac Newton* and him, become infinite, when the orifice is equal to the breadth of the veſſel; which ſhews a plain contradiction, to all experience. For according to their rules, the water may acquire any velocity whatſoever in its deſcent; whereas it is manifeſt, that it never can have a greater than that acquired by a body falling through the ſame height: Since whatever the width of the veſſel is, it is evident that the orifice LM is only preſſed by a column of the ſame baſe and which has the ſame altitude as the water, and in its deſcent its motion will be retarded by the lateral preſſion of the ambient water; it is likewiſe plain from all the experiments that ever were made, that the water never runs out with as great a velocity thro' an orifice, than through a veſſel without any bottom; and therefore their rules can never be true in general, whatever they may be in any particular caſe.

THEOREM III.

Fig. 17. 599. *If to an upright veſſel* ABCD, *there be joined a vertical tube* LMNO, *and in* OL *produced upwards, the point* K *be taken in ſuch a manner, that a body falling through the height* KL *acquires the velocity with which the water runs into the tube,* KO *will be the height through which a falling body acquires the velocity of the water at the orifice* ON.

**Art* 586. For it has been proved * that the velocity acquired by the water in the deſcent through a tube is equal to that acquired by a body falling thro' the ſame height. Therefore the velocity of the water, acquired by deſcending from the veſſel through

through the tube, is equal to that of a body falling thro' the height KO.

THEOREM IV.

666. *If to an upright veſſel* ABCD, *an horizontal tube* DEFG *is joined; the velocity with which the water runs out at the orifice* GF, *will be as the ſquare root of the difference between the height* EK *through which a falling body acquires the mean velocity with which the water enters the pipe, and a part of the tube* DG, *which is in a conſtant ratio to the whole length.* Fig. 18.

For it is evident that the water would run out of the orifice DE, in the ſame manner as if it were in the bottom, if the water in the tube DF did not retard its motion; but this obſtacle is in proportion to the quantity of water in that tube, or becauſe it is of an equal bigneſs, as the length DG; and becauſe the motion of a body falling through KE, is every where retarded, the velocity at the orifice ED will, by courſe, be leſs than if there was no obſtacle; and therefore the velocity at the orifice GF, is as the ſquare root of the difference between the height EK, and a part of the length DG of the tube, which is in a conſtant ratio of the whole.

The reſiſtance of the motion in the horizontal pipe is not only owing to the quantity of water in it, but likewiſe to the friction of the water againſt its ſides; but as this reſiſtance is likewiſe proportional to the length of the pipe, the whole reſiſtance which retards the motion of the water, will ſtill be proportional to the length of the pipe.

In the conduct of water from one place to another, where the pipes aſcend and deſcend, wind and turn about, it is impoſſible to give any certain

tain rule for the quantities of water discharged, for which reason we shall insert here some of the best experiments we could get, which may be of some use upon such occasions; they were made by Mr. *Couplet*, member of the royal academy of sciences at *Paris*.

A pipe that goes from *La Place Dauphine*, at *Paris*, to *Versailles*, made of iron, of 4 inch diameter, and 297 toises long; the fall of the water being 9 inches, discharged 896 cubic inches of water in 34.5 seconds, which is 26 inches in a second: The fall of the water being 21 inches, the quantity of water discharged was 896 inches in 20 seconds, or 44.8 in one second; and when the height of the water was 31 inches, it discharged 57.8 inches in a second.

In order to find the quantities discharged according to the theory, we must substitute the respective values for the orifice $\odot$, and the height b into $2 \odot \sqrt{180b}$; and because the diameter of pipe is 4 inches, its area will be 12.5, which gives $25\sqrt{180b}$, and by substituting the halves * of 9, 21, 31, for b, we shall have

* art. 591.

discharged by experiment.	*discharged by theory.*	*differences.*
26	712.5	686.5
44.8	1087.5	1043
57.8	1318.7	1261

If we suppose the differences to be proportional to the velocities, as they ought to be, since all the rest is the same, we shall find 686.5, 1046, 1273, which differ from the differences above by .0003, .0009.

The

The second experiment was made with a pipe of 6 inches diameter, and 285.5 toises long; the fall of the water being 3 inches, the discharge was 77.9 inches in a second; and when the fall was 5.25 inches the quantity discharged was 112. Now because the diameter of the pipe is 6 inches, its area will be 28.3; and hence $2 \odot \sqrt{180b}$, becomes $56.6\sqrt{180b}$; in which substituting half of 3, and 5.25, for b, we shall have 928, and 1228, for the quantities discharged by the theory; and the differences are 849 and 1116, which are nearly as the velocities, for there wants but .0007.

Two pipes of 600 toises long, the diameter of the one was 18 inches, and that of the other 12; the height of the fall was 144.5 in both; the first discharged 6965 inches in a second, and the second 2656 inches in the same time. These quantities are not proportional to the sections of the pipes, as they should be, since the fall of the water and lengths of the pipes are the same in both; this may be occasioned by the different turns and windings of the pipes,

The same gentleman made two others, one with an iron pipe of 18 inches diameter, and 790 toises long, which discharged 3688 inches, the fall being 55.5 inches; the diameter of the other was 12 inches, and 2340 toises long, it discharged 1792 inches in a second; but according to the theory the first should have discharged 75256, and the second 33335.

To these experiments we shall add some observations, useful in the conduct of water. In pipes which rise and fall, the air that gets in with the water, rises to the highest parts and there it remains, whereby the motion of the water is much

retarded. To remedy this, ſmall vertical pipes are inſerted in thoſe places, with cocks to let out the air when the motion of the water begins to diminiſh. It happens likewiſe that when the water in the reſervoir is muddy it leaves dirt in the pipes, and ſometimes choaks them up; in ſuch a caſe bellowſes are uſed to force them, by which they are cleared again.

THEOREM V.

Fig. 19. 601. *If a tube* EFGHI *be fixed to the bottom of a reſervoir* ABCD, *and the point* K *be ſo taken in the line* EF *produced upwards, that a body falling through the height* KE *may acquire the velocity with which water runs into the tube; then the water will run out of the orifice* I *with a velocity equal to that of a body falling through the height of the point* K *above the orifice.*

For the velocity of the water in the loweſt part F, will be equal to that acquired by a body falling through KF; and ſuppoſing the horizontal part FG of the pipe to be but ſhort, it will riſe in the vertical part GH, with the velocity at F, and loſe as much of its velocity in the aſcent * a ſa body would acquire by falling through the height HG. If therefore the horizontal line LH be drawn, the water will loſe juſt as much of its velocity in the aſcent through GH, as it acquired by deſcending through LF, and conſequently will run out of the orifice L, with the velocity acquired by a body falling through the height KL of the point K above the orifice.

art. 453

COR.

602. Hence it is evident, that if the vertical part GH is but ſhort, and the orifice upwards, the

the water will rise nearly to the same height with the point K; for the resistance of the air, and the friction in the pipe, will diminish the height of the water by some small matter.

Those who made experiments on jets of water have neglected the dimensions of the cysterns or reservoirs, so that it is not possible to know how much the rise will be less than the fall; they all imagined that it should be as high as the surface of the water, because they took that height to be that through which a falling body acquired the velocity of the water; experience has, indeed, shewn, that it does not rise so high; but the defect was attributed to the resistance of the air and that of the tube; though the difference between the rise and fall of water is not so much, considering the true height of the fall; yet it is certain that the resistance in the tube, and that of the air, diminishes the rise in high jets, very considerably. It is a common thing to fix a short brass tube to the orifice, of about an inch long, and of a lesser diameter than the other pipe, or only a thin plate of brass with an opening; this last is esteemed the best, on account, as it is said, that the resistance is almost nothing through this orifice, and to make it rise to its full height the jets are somewhat inclined from the vertical, so as hardly to be perceptible by the eye, that the water above may not fall back upon that which rises, and hinder its motion.

The pipe through which the water descends, is made larger than that through which it ascends; but the proportion is not certain, and is chiefly guessed at by the writers of experiments.

Mr. *Belidor* will have it * that the velocity with

* *Architecture hyd. vol* 2. *book* 4. *art.* 1225.

which the water runs out at the orifice I, is to be expreſſed by the difference of the ſquare root of the deſcent KF, and that of the aſcent GI; contrary to all other authors, and to what we have proved in the laſt theorem; and he thinks, *art.* 1211, that when the aſcending part of the tube GH is the $\frac{4}{9}$ of KF, the water will run out of the orifice with a greater velocity than if it were of any other; but this is as erroneous as the former; for it is plain, even according to his own principles, that the leſs the height GH is, the greater that velocity will be, and therefore when GH is very ſmall, or nothing, the velocity at the orifice will be the greateſt of all, ſince it is then equal to that acquired by a body falling through the height KF.

PROB. II.

603. *To find the mean velocity of the water running out of a rectangular orifice* ABCD, *made in the ſide of a veſſel.* Fig. 20.

Let LE be the height through which a falling body may acquire the velocity of the water at the bottom AD; then will LF be that whoſe ſquare root will expreſs the velocity at BC; and if MN be drawn any where parallel to AD, interſecting LE in P, the ſquare root of LP will expreſs the velocity of the water in the ſection MN; and therefore the ſum of all the velocities of the different ſections will be equal to the area whoſe baſe is EF, and the ordinate or perpendicular in any given point P, is always expreſſed by the ſquare root of the correſponding line PL.

Now if $EL = a$, $FL = b$, the variable quantity $PL = x$, and the correſponding ordinate y; then will $\sqrt{x} = y$, by what has been prov-

ed

ed before, or $x = yy$; but this is the equation * of a parabola, whoſe parameter is unity; and therefore $\frac{2}{3}xy$, or $\frac{2}{3}x\sqrt{x}$ will expreſs the area * corresponding to the abſciſſa LP; and when LP becomes LE, or $x = a$, that area becomes $\frac{2}{3}a\sqrt{a}$, but when LP becomes FL, or $x = b$, it becomes $\frac{2}{3}b\sqrt{b}$; therefore the difference $\frac{2}{3}a\sqrt{a} - \frac{2}{3}b\sqrt{b}$, will expreſs that part of the area correſponding to the baſe EF, or the ſum of all the velocities of the water in the different ſections.

* art. 256.

* art. 275.

Now if $EF = c$, and h be the height through which a falling body may acquire the mean velocity, then will $c\sqrt{h} = \frac{2}{3}a\sqrt{a} - \frac{2}{3}b\sqrt{b}$, by ſuppoſition; by ſquaring both ſides, and dividing by cc, we ſhall have $h = \frac{4}{9} \times \frac{a^3 + b^3 - 2ab\sqrt{ab}}{cc}$.

Which ſhews that this height is equal to $\frac{4}{9}$ of the ſum of the cubes of EL *and* FL, *minus two parallelepipedons, made of theſe lines and a geometrical mean between them, divided by the ſquare of the height* EF *of the orifice.*

When the height EF of the orifice is but ſmall, in compariſon to the height EL, then h may ſafely be taken equal to the diſtance from the center or middle of EF, to the point L. For if $a = 12$, $b = 10$, and $c = 2$; then will $h = 11.1$; which is nearly $= \frac{a+b}{2} = 11$.

The mean velocity in a circular orifice is very difficult to find, for which reaſon we ſhall omit it here; neither is there any uſe for large circular orifices in practice; for the holes made in the gates of ſluices to let out the water, are always

rectangular, as likewise the passages which lead the water to mills.

Of BODIES *moved by* FLUIDS.

THEOREM VI.

Fig. 21. 604. *If a plane surface* AB *be perpendicular to the stream of any fluid, the impression of the fluid on this plane will be in the compound ratio of the square of the velocity and the surface pressed.*

For with twice the velocity the impression of each particle will be double, and twice the number of particles will strike the plane in the same time, setting aside all impediments, and with a triple velocity, the impression of each particle will be triple, and three times the number of particles will strike the plane in the same time: Therefore the impression will be quadruple with twice the velocity, nintuple with thrice the velocity. By the same way of arguing, the impression will always be found to increase, in the same proportion as the squares of the velocities: And because the surface of the plane expresses the number of particles striking at the same time; it follows that the sum of all the particles striking the plane in any given time, or the impression of the fluid on the plane, is in the compound ratio of the square of the velocity of the fluid, and the area of the plane.

COR. I.

605. Hence, if the surface AB be oblique to the direction *c a* of the stream, by drawing *cf* perpendicular to the plane, and joining *df*; then because the force in the direction *c d* is * equivalent

*art. 446.

to

to the forces in the directions df parallel and cf perpendicular to the plane, and the impreſſion of the fluid on the plane is made according to the latter; if cd expreſſes the velocity of the fluid, the ſquare of cd will be to the ſquare of cf, as the abſolute force of the fluid is to the relative part which impreſſes the plane. Conſequently, the ſquare of the radius cd is to the ſquare of the ſine cd of the incident angle, as the abſolute force of the fluid is to the part with which it ſtrikes the oblique plane.

COR. II.

606. Hence, becauſe the abſolute force of the fluid with which it would ſtrike the plane AB in a perpendicular direction, is as the product of that plane and the ſquare of cd; the relative force of the fluid with which it ſtrikes that plane in a direction oblique, will be as the product of that plane, and the ſquare of the ſine cf of the incident angle cdf.

COR. III.

607. If a fluid ſtrikes a plane which is in motion, in a perpendicular direction, it is evident that each particle ſtrikes that plane with a velocity equal to the difference between * the velocity of the fluid and that of the plane; and conſequently the impreſſion of the fluid on that plane, will be as the ſquare of the difference between the velocities of the fluid and that of the plane.

* art. 465.

PROB. III.

608. *To find the greateſt momentum poſſible of a plane ſurface moved by a fluid.*

Let a expreſs the conſtant velocity of the fluid

x, the difference between the velocities of the fluid and that of the plane; then becauſe xx, ex-
*art. 607. preſſes * the force of the fluid againſt that ſurface, and this force multiplied by the velocity $a-x$ of
*art. 480. the ſurface gives $axx-x^3$ for * its momentum; it is evident that this expreſſion $axx-x^3$, ought
* art. 542. to be the greateſt poſſible, which is when * $3x=2a$. Conſequently, when the velocity $(a-x)\frac{1}{3}a$ of the plane is one third of the velocity of the ſtream, it will have the greateſt momentum poſſible.

COR. I.

609. Hence, becauſe xx expreſſes the force of the fluid, which becomes $(xx=)\frac{4}{9}aa$, when it produces the greateſt momentum poſſible. If
Fig. 20. ABCD be the pallet of a wheel whoſe axis is TT, which moves an engine; and this engine being charged with ſuch a weight P, as will be juſt in equilibrio with the preſſure of the water; it is evident that $P=xx$; and when the engine has the greateſt momentum poſſible, we have $P=\frac{4}{9}aa$. Conſequently, when the engine is firſt charged with a weight, ſo as to remain in equilibrio, and then with the $\frac{4}{9}$th of that weight only, it will produce the greateſt effect poſſible.

When a plane ſurface is moved by a fluid, and turns about an axis, ſuch as the pallets of a water wheel, there is a point in that plane, which being ſtruck by the whole force of the fluid, will produce the ſame effect, as if every part of that plane were ſtruck by its reſpective force. This point is called the Center of Percuſſion: I have given a general method for finding this center in my treatiſe of conic-ſections, page 159; but as it depends

pends on the method of fluxions, we shall investigate that center here in a rectangular figure, which is the only one wanted.

PROB. IV.

610. *To find the center of percussion of a rectangle* ABCD, *turning about the axis* TT, *perpendicular to the plane of the figure.* Fig. 20.

If we consider the velocity with which the fluid strikes any line MN parallel to the axis TT, of rotation, as a weight applied to the end P of the leaver PL, endeavouring to turn that line about the axis TT; it is evident that the momentum of this weight is as * the product PL multiplied by that weight or velocity of the point P; but the velocity of that point is as the arc described in the motion of the rectangle about the axis TT; and the arcs described by the points P, E, F, are as the radii PL, EL, FL, and therefore the momentum of the point P, is as the square of PL, and the momentum of the whole rectangle ABCD as the area, whose base is EF, and ordinate or perpendicular as the squares of the respective distances PL from the axis TT. *art. 480.

Whence, if $EL = a$, $FL = b$, and the indeterminate quantity $PL = x$; then will xx express the ordinate at the point P; and hence the area corresponding to the base PL will be * $\frac{1}{3}x^3$. Now when PL becomes EL, or $x = a$, we shall have $\frac{1}{3}a^3$ for the area of the base EL; and when PL becomes FL, or $x = b$, we have $\frac{1}{3}b^3$ for the area of the base FL; therefore the difference $\frac{1}{3}a^3 - \frac{1}{3}b^3$, expresses the area of the base EF, or the momentum of the figure ABCD. *art. 233.

And becauſe the velocity of any point P has been proved to be as PL (x), the ſum of all the velocities of the different parts of the figure, or of the line EF, will be equal to the area of a figure whoſe baſe is EF, and perpendicular in any point P, as the reſpective diſtance PL; but the area of
*art. 233. the baſe PL will be * as $\frac{1}{2}xx$, which when PL becomes EL, or $x=a$, will be $\frac{1}{2}aa$; and when PL becomes FL, or $x=b$, $\frac{1}{2}bb$; it follows that $\frac{1}{2}aa-\frac{1}{2}bb$ expreſſes the area of the baſe EF, or the ſum of the velocities of the rectangle. Now if the diſtance required be d, then $d\times\overline{\frac{1}{2}aa-\frac{1}{2}bb}=\frac{1}{3}a^3-\frac{1}{3}b^3$, by ſuppoſition, or dividing by $a-b$, $\frac{1}{2}d\times\overline{a+b}=\frac{1}{3}\times\overline{aa+ab+bb}$; and dividing again by $\frac{1}{2}\overline{a+b}$, $d=\frac{2}{3}\times\overline{a+b}-\frac{ab}{a+b}$ which gives this

GENERAL RULE.

Add the diſtances of the points E *and* F *from the axis of rotation, and ſubtract the fourth proportional to the ſum of theſe diſtances, and to the diſtances taken ſingly, from the ſum, then the two thirds of the difference will give the diſtance required.*

PROB. V.

Fig. 22. 611. *To find the angle which the rudder of a ſhip* AE *ought to make, ſo as to turn the ſhip with the greateſt ſwiftneſs poſſible.*

Let LA be the direction of the ſhip, draw LD perpendicular to the rudder, and DH to the direction LA; then the ſquare of LA is to the
*art. 605. ſquare of LD * as the abſolute force of the ſtream

is

is to the force with which it ſtrikes the rudder, in the direction LD; but this force is equivalent to the force DH perpendicular, and to LH parallel to the ſtream; and therefore ſtrikes the rudder by the part DH, ſo as to make it turn.

If $LD = a$, and $DA = x$, then will $\overline{LD}^2 = aa - xx$, expreſs the force in the direction LD; and LD is to DH, or becauſe of the ſimilar triangles LHD, LDA, $LA(a) : AD(x)$ as the force $aa - xx$ in the direction LD, is to the force in the direction DH, which therefore is $= \frac{aax - x^3}{a}$; and when * $3xx = aa$, that force will be the greateſt poſſible. Conſequently a is to $a\sqrt{\frac{1}{3}}$ as the radius 100000 is to the ſine AD $= 57733$, of the complement ALD; this ſine anſwers to an angle of 35 degrees and 16 minutes. Conſequently the angle LAD ſought will be 54 degrees and 44 minutes.

* art. 541.

PROB. VI.

612. *To find the angle which the plane of the ſails* E, F, *of a windmill ought to make with the axis* BA, *ſo that a wind blowing uniformly in a direction ab parallel to the axis, ſhall make it turn with the greateſt ſwiftneſs poſſible.* Fig. 23

Let LBD be the angle required, from any point L in the axis produced, draw LD perpendicular to DB, and DH to LB; then the ſquare of LB is to the ſquare of LD, as the abſolute force of the wind to the force with which it ſtrikes the ſails in the direction LD; but this force is equivalent to the force DH perpendicular, and to the force LH parallel to the direction LB of the wind; and therefore the wind will move the ſails with

with the force in the direction DH only. But LD is to DH, or LB to BD, as the force $\overline{LD}^2$ in the direction LD to the force in the direction DH.

If LB$=a$, DB$=x$, that force will be $\frac{aax-x^3}{a}$, which being the same as that in the former problem, gives the same number of 54 degrees and 44 minutes for the angle LBD required.

PROB. VII.

Plate XXIV. Fig. 24. 613. *To find the resistance of a right cone* ACK, *moving in a fluid in the direction* AB *of its axis with its vertex* A *foremost.*

Let the line LEF be parallel to the axis AB, from any point L draw LD perpendicular to the side AC of the cone, and DH to LE; then if LE expresses the absolute force of a particle, LD will be the relative part with which it strikes the surface in E; and because the force in the direction LD is equivalent to the forces in the directions DH perpendicular, and LH parallel to the direction of the cone, the former being destroyed by some opposite force, the latter will be that by which that particle resists the motion of the cone. Now because of the similar triangles EDL, ABC, we have AC:CB::LE:LD; and the similar triangles ABC, DHL, give AC:CB::LD:DH; if the terms of these two proportions are multiplied together, we shall have $\overline{AC}^2 : \overline{CB}^2 :: LE : LH$. Therefore the force with which a particle of the fluid would resist the base KC is to that with which it resists the surface of the cone in the given ratio of the square of AC, the side of the cone, to the square of

BC

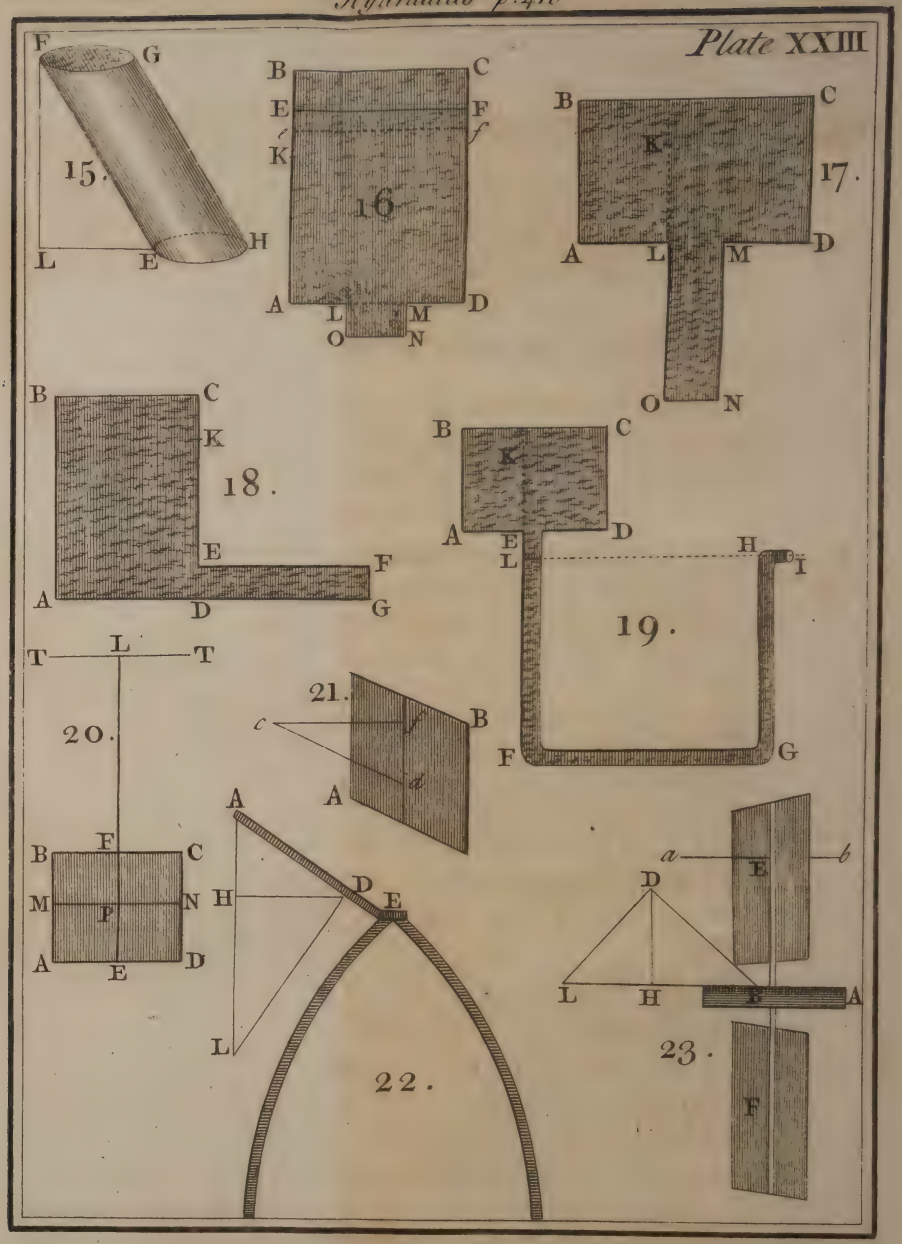


Plate XXIII
15.
16.
17.
18.
19.
20.
21.
22.
23.

BC the radius of the base; and as the number of particles which at a time can strike the base of the cone is equal to the number which at a time can strike the surface; the resistance of the base is to that of the cone as the square of the side AC is to the square of the radius CB of the base.

PROB. VIII.

614. *To find a frustrum of a cone of a given base* CK *and altitude* DB, *which moving in the direction of the axis, with the smaller base foremost, shall meet with the least resistance possible.* Fig. 25.

Let the sides CE, KF, produced, meet the axis in A; and let EF be the base of the cone required, by drawing DI, BG, perpendicular to the side AC; then because the resistance of the base CK, is to that of the conic surface ACK, as the square of AC is to the square of CB; or because of the similar triangles ABC, BGC, as the square of CB is to the square of CG; and since CB is given, the resistance of the conic-surface will be as the square of CG. By the same way of reasoning the resistance of the part wanted AEF, is to that of the base EF, as the square of IE to the square of DE; and therefore the difference between the resistance of the base EF, and that of the cone AEF, is as the difference between the squares of DE and IE; that is, as the square of DI.

Now if from the point D, a line DH be drawn parallel to the side AC, intersecting BG in L; then because DI is equal to LG, the resistance of the frustrum ECKF, will be as the sum of the squares of LG and GC; or by drawing CL meeting the axis in O, that resistance is as the

square

ſquare of CL. Whence, if upon the altitude DB of the fruſtrum, as diameter, there be deſcribed a ſemi-circumference of a circle; then becauſe the angle DLB is a right one, the point L will be in the circumference; and when the line CL produced paſſes through the center, it will
*art. 121. be * the ſhorteſt of all the lines that can be drawn from the given point C to the circle. Conſequently, if OA be taken equal to OC, the point A will be the vertex of the fruſtrum required. For ſince DL is parallel to AC, and OD = OL, we ſhall have OA = OC.

This ſolution has been communicated to me by Mr. *George Campbell*, ſtorekeeper to the office of ordnance at *Woolwich*.

PROB. IX.

Fig. 26. 615. *To find the reſiſtance of a ſphere* ATBR, *moving in a fluid according to the direction of the diameter* AB.

Draw any where the line LP perpendicular to the diameter RT, interſecting the circumference in M; and from any point L in that line draw LD perpendicular to the tangent MD at M, and DH to LP; then if LM expreſſes the abſolute force of a particle, LD will expreſs the relative part, with which it reſiſts the ſphere in the direction LD; but this force is equivalent to the forces in the direction DH perpendicular, and to LH parallel to that of the ſphere; the former being deſtroyed by a force in a contrary direction, and the latter is that with which that particle reſiſts the motion of the ſphere. Now if the radius CM be drawn, the ſimilar triangles CPM, LDM, give CM:PM::ML:DL, and the ſimilar triangles CPM, LHD, CM:PM::DL:LH; if the terms

terms of these two proportions are multiplied together, we shall have $\overline{CM}^2 : \overline{PM}^2 :: ML : LH$.

Now if the curve FCE be such that by producing MP so as to meet that curve in N, and EF parallel to RT, in Q; QN be always a third continued proportional to the radius CM and the line PM; then will *$CM : QN :: \overline{CM}^2 : \overline{PM}^2$; and therefore $CM : QN :: ML : LH$, by equality of ratios; that is, the absolute force MI, with which a particle resists the circular base EF is to the force HL, with which that particle resists the sphere in M, as the radius AC, or PQ, is to QN; therefore the force with which the line EF is resisted, is to the force with which the semi-circle RBT is resisted as the rectangle ETRF is to the area ENCF, of that curve-line figure. And the whole resistance against the circular base FE, is to the whole resistance of the sphere as the cylinder FT, described by the rectangle AT, is to the solid described by the space ACNE, about the axis AC. * art. 76.

Now because $AC : PM :: PM : QN$, by construction, and * RP, or $FQ : PM :: PM : PT$, or QE, by the property of the circle; therefore *$AC : FQ :: QE : QN$; that is, if any line QN be drawn parallel to AC, the rectangle of the parts FQ, QE, terminated by that line, is equal to the rectangle of that line and a given one AC; which is the property * of the parabola; and as the content of the paraboloid is * half the circumscribed cylinder FT, it follows that the resistance of the sphere is half the resistance of the circumscribed cylinder. * art. 117. * art. 85. * art. 231. * art. 276.

SECT.

SECT. VII.

Of PNEUMATICS.

AS air has a conſiderable ſhare in the effects of pumps, ſyringes, fire-engines, barometers, and eſpecially in the effects of gunpowder, it is neceſſary to explain here, in a few propoſitions, its principal properties, as far as they are uſeful, leaving thoſe which are more curious than neceſſary in machines, to other authors, our chief deſign being to write for the practical reader.

THEOREM I.

616. *If all the particles of air have the ſame elaſtic force, the force of compreſſion is as the denſity.*

For ſince the particles are all endued with the ſame elaſtic force, by ſuppoſition, the compreſſing force muſt be as the number of particles preſſed; but equal bulks of air, whoſe denſities are twice, three times, four times, greater than that of another of the ſame bulk, contain * twice, thrice, four times the number of particles; and in the ſame manner it is proved, that the denſity is always proportional to the number of particles preſſed. Therefore the force of compreſſion is as the denſity.

*art. 572.

COR. I.

617. Hence, ſince the quantities of matter contained

tained in a body is as * its denſity and magnitude conjointly; if the ſame quantity of air be contained in different ſpaces, the denſities are reciprocally as the ſpaces in which they are contained; and ſince we have proved that the compreſſing force is as the denſity, it follows that the compreſſing forces are reciprocally as the ſpaces, which contain the ſame quantity of air.

* art. 571.

COR. II.

618. Hence, if a cylinder ADHE is filled with air, and that quantity of air be reduced to the ſpace ACGE, or to ABFE, the force which retains the air in the ſpace ADHE, is to the force which retains it in the ſpace ACGE, or ABFE, as the height AC or AB is to the height AD of the cylinder: For theſe ſpaces having the ſame baſe AE, are to each other as their altitudes.

Fig. 27.

COR. III.

619. If AK be perpendicular to AD, and with the aſymptotes AD, AK, there be deſcribed an equilateral hyperbola LMN, by drawing any where the lines BL, CM, DN, perpendicular to AD; then the compreſſing forces which reduce the ſame quantity of air into the ſeveral ſpaces ADHE, ACGE, ABFE, are as the correſponding perpendiculars DN, CM, BL. For becauſe AD : AB :: BL : DN, by the * property of the hyperbola; and AD is to AB * reciprocally as the compreſſing forces at BF to the compreſſing force at DH; we have BL is to DN as the compreſſing force at BF is to the compreſſing force at DH, by equality of ratios.

* art. 351.

* art. 618.

COR.

COR. IV.

620. Since the rarity of an homogeneous fluid is reciprocally as the denſity, and AD : AB :: BL : DN; by the above noted property of the hyperbola, the rarity of the ſame quantity of a fluid contained in ABFE is to its rarity when contained in ADHE, as AB is to AD.

COR. V.

621. Since AB decreaſes as BL increaſes, it is evident that when AB becomes indefinitely ſmall, BL becomes indefinitely great; and therefore an elaſtic fluid may be compreſſed at infinitum, provided a ſufficient force could be found to do it.

PROB. I.

622. *To find how much the air is rarified in a receiver, after any given number of turns.*

Let the capacity of the receiver be to that of the barrel of the pump, as c to 1; then becauſe the air will be rarified by any one of the turns in the ratio of the ſpace c to the ſpace $1+c$; that is, it will be rarified $\frac{1+c}{c}^2$ times in two turns, $\frac{1+c}{c}^3$ in three, and, in general, $\frac{1+c}{c}^n$ times in n turns.

If the capacity of the barrel be equal to that of the receiver; that is, if $c=1$, then will 2^n expreſs the number of times that the air is rarified. If for example, $n=10$, then will $2^n=1024$. Hence, in ten turns the air is 1024 times rarified.

COR.

623. Hence, if it be required to find the number

ber of turns, so as to rarify the air any certain number r of times; then will $\overline{\frac{1+c}{c}}\Big|^n = r$; or by the noted property of logarithms, we have $nl\frac{1+c}{c} = lr$; or, $nl\overline{1+c} - l.c = lr$. Hence $n = \frac{lr}{l1+c-lc}$.

Examp. Let $r = 100$, $c = 1$; then will $n = \frac{l\,10}{l\,2} = 6.6$; so that in six turns, and six-tenth part of a turn, the air will be 100 times rarified.

REMARK.

It is well known, that gun-powder inflamed is nothing else but an elastic fluid, and therefore the greater the quantity of this fluid is contained in the same space, the more violent the explosion will be; so that if the proportion between two different quantities of inflamed powder was known, the force which they communicate to a cannon ball, would be easily determined; but as powder is never lighted all at once, contrary to the supposition of a late author, when confined in pieces of artillery, it is impossible to measure these forces exactly, as long as the times of inflamations are not known: For it has been found by many experiments, as well as by daily practice, that there is always some unlighted powder blown out of the pieces; it is true the abovementioned author imagined that the powder blown out unfired, was of an inflamable substance, but this opinion has been sufficiently refuted by the gentlemen of the Royal Society, who having gathered together as much of it as loaded a pistol, and which being discharged, went off as before.

There was not the leaſt occaſion to make any experiments to prove the contrary of this ſuppoſition; ſince it is manifeſt, that as powder lights gradually, when there is more than can fire during the time the ſhot moves out of the piece, the overplus muſt, by courſe, be blown out unfired.

Thoſe who pretend to improve gunnery by the help of this principle, are groſly miſtaken; for all powder is not alike ſtrong, ſome being better made than others, and ſometimes it is ſpoiled by lying too long in damp places, not even in the ſame barrel; becauſe that which is near the ſides of the caſk loſes a part of its ſtrength by the affinity of the air, which penetrates the pores of the wood, and the ſaltpetre being heavier than the other ingredients, when the barrel lies for a time on the ſame ſide, it deſcends, whereby the upper part of the powder becomes weaker.

But granting that the powder is equally ſtrong in all ſeaſons, there can no concluſion be drawn from thence, ſince the different manner in which it is placed in pieces, is ſufficient to overthrow all the principles on which its force depends; as for example, in a mortar where the chamber prevents the ſhell from lying cloſe to the powder when it is not quite full, it is viſible that it cannot act in the ſame manner as when full; and therefore no principles can be eſtabliſhed in reſpect to the force which powder communicates to the ſhells.

THEOREM II.

624. *If a glaſs tube, of about three feet long, be filled with quickſilver, and its orifice be covered with a finger; then by inverting the tube, and immerſing the end ſtopped with the finger in a veſſel*

filled with quickſilver, upon the removal of the finger from the orifice, the quickſilver in the tube will deſcend till its weight is in equilibrio with that of the ambient air.

For becauſe the upper end of the tube is ſtopped, the air cannot preſs upon the quickſilver in the tube, but preſſing the ſurface of that in the veſſel, with a force equal to its own weight; and the quickſilver in the tube preſſes alſo that ſurface by its whole weight; it follows neceſſarily, that the weight of the quickſilver in the tube muſt be in equilibrio with that of the ambient air.

COR. I.

625. Hence it is manifeſt, that the weight of the quickſilver in the tube, above the ſurface of that in the veſſel, is equal to the weight of a column of air of the ſame baſe of the tube, and whoſe altitude is equal to the height of the atmoſphere.

COR. II.

626. It is alſo evident, that the weight of the air varies at different times; for the quickſilver in the tube is found to riſe and fall from the height of 31 inches to 28, which variation muſt be owing to the different preſſures of the air on the ſurface of the quickſilver in the veſſel, ſince the weight of the quickſilver remains the ſame.

N. B. It is upon this principle that the barometers and other weather-glaſſes are made: For a barometer is nothing elſe but a glaſs tube of about 32 inches long, inverted into a veſſel filled with quickſilver, and the interval between 31 and 28 inches, from the ſurface of the quickſilver in the veſſel, being divided into any number of equal

parts, by which the rise and fall of the quicksilver in the tube, and, consequently, the different pressures of the air, are distinguished.

COR. III.

627. Hence, the weight of a column of air of any given base may be computed. For since a cubic foot of quicksilver weighs 14000 ounces, or *art. 580 875 pounds, according to the tables *, when the quicksilver stands at the height of 30 inches, or 2.5 feet; a column of quicksilver, whose base is a square foot, and its altitude 2.5 feet, will weigh 2.5×875, or 2187.5 pounds; which is therefore also the weight of a column of air at that time, standing on a base of a square foot.

The surface of a middle-sized man is computed to be about 15 square feet; and therefore the weight of the air which presses such a surface will be 15×2187.5, or 32812.5 pounds.

From whence it seems to be surprizing that we should not perceive this great pressure; but it must be considered, that we are every where pressed alike, by which it is felt no where; whereas if it was taken away in any part, it would be so great as we should be hardly able to bear it; for by laying a hand on the mouth of the air pump, in a few turns the pressure appears so great, that if more air was pumpt out, it would break the hand.

THEOREM III.

628. *If a tube be filled with water, or any other fluid, and immersed into a vessel full of the same fluid, with the open end downwards; the height to which that fluid will rise in the tube, will be to the height of the quicksilver in the barometer at that time, reciprocally as the specific gravity of the quicksilver is to the specific gravity of the fluid.* For

For becauſe both the fluid and the quickſilver in the tubes being in equilibrio with the ſame weight of the ambient air, their weights muſt be equal to each other; but the weights of bodies are in the compound ratio * of their denſities and magnitudes, and the columns of the fluid and the quickſilver, which have equal baſes, are as their altitudes, and the ſpecific gravities * as the denſities: It follows that the height of the fluid in the tube is to the height of the quickſilver in the barometer, as the ſpecific gravity of the quickſilver is to the ſpecific gravity of the fluid.

*art. 571.

*art. 573.

COR. I.

629. Hence, ſince the ſpecific gravity of water is to that of quickſilver as unity to 14; when the quickſilver ſtands at 30 inches high, if we ſay, 1 : 14 :: 30 : 420, this fourth term will expreſs the height to which water will riſe in a tube at that time, expreſſed in inches; or the water will riſe to the height of 35 feet at that time: But if the quickſilver ſtands at 28 inches, then the water will only riſe to the height of 32.5 feet.

COR. II.

630. Hence, water can be raiſed no higher than about 34 feet, by means of a ſucking pump; becauſe this pump draws the air out of the pipe, by which the preſſure of the ambient air forces the water to riſe to ſuch a height, as to be in equilibrio with that preſſure, which has been proved to be about 34 feet at a medium; and therefore the water cannot riſe higher. For which reaſon, if it be required to riſe it to a greater height the forcing pump muſt be uſed.

PROB. II.

Fig. 28. 631. *If a tube be filled with quickſilver, or any other incompreſſible fluid whatſoever, all but a part ef, which remains filled with air; when the tube is inverted, and its orifice immerſed in a veſſel filled with the ſame fluid, it is required to find the height bc to which the fluid will riſe.*

Let *bd* be the height to which the fluid would riſe, if no air was included in the tube; then the preſſure of the atmoſphere upon the ſurface of the veſſel would be * balanced by a column *bd* of the fluid; but here it is balanced by the weight of the column *bc*, and the preſſure of the included air in *ce* downwards; the weight of the column *bd* is therefore equivalent to the weight *bc*, and to the elaſtic force of the air in *ce*. Whence, if the column *bc* be taken away, which is common to both, the weight of the fluid in *cd* will be equivalent to the elaſtic force of the air in *ce*; but when the air is in its natural ſtate, it ballances the weight of the fluid in *bd*; and this weight is to that in *cd*, as *bd* to *cd*; and by the elaſticity of the air, the force, when confined in *ef*, is to the force when confined in *ce* * as *ce* is to *fe*. Conſequently, $bd : cd :: ec : ef$, by equality of ratios.

* art. 625

* art. 618.

If *de* be biſected in *g*, and we ſuppoſe $bd = b$, $ef = c$, *ge* or $gd = a$, and $cg = x$; then will $cd = x - a$, $ec = x + a$; and by ſubſtituting theſe values into the proportion above, we get $b : x - a :: x + a : c$, or, $bc = xx - aa$; therefore $x = \sqrt{aa + bc}$.

Examp. I. Let the length *be* of the tube be 35 inches, and the height $bd = 30$; then will $ef = 5$, and

and $gd = 2.5$; whence, by making $a = 2.5$, $b = 30$, $c = 5$, we get $cg = \sqrt{6.25 + 150} = 12.5$, and $gc - gd = 12.5 - 2.5$; or, $cd = 10$, and $bc = 20$.

Examp. II. Let the height be of the tube be 36 inches, $bd = 30$, and let the quantity of air ef be required so as the quicksilver may raise 18 inches; then because $de = 6$, $gd = 3$, $cd = 30 - 18 = 12$, we have $a = 3$, $x = 15$, $b = 30$; and therefore the equation $bc = xx - aa$, gives $c = \frac{xx - aa}{b} = 7.2$ inches, for the height ef of the air required.

When the length of the tube is less than the height to which the fluid would rise; the same equation will still hold good. For if the length of the tube $be = 23$ inches, $ef = 2$, and the height bc to which the fluid will rise; then by supposing $bd = 30$, we get $bd - be = de = 7$, $ge = a, = 3.5$, $b = 30$, and cg, or $x = \sqrt{12.25 + 60} = 8.5$; hence $cg + gd = cd = 12$, and $bc = 18$.

PROB. III.

632. *To find the altitude to which the quicksilver will rise when there is a given quantity of any other fluid in the same tube.* Fig. 29.

It is evident that the sum of the weights of the quicksilver and the fluid, is equal to the weight of the ambient air, or to the column of quicksilver, which is equal in weight to the air: If therefore the height of the column of quicksilver, equal to the column of the fluid, be subtracted from the column of quicksilver which ballances the weight of the air, the difference will be the

height to which the quickſilver will riſe in the tube, being charged with that quantity of the fluid.

If a expreſſes the height of the fluid x that of quickſilver, whoſe weight is equal to that of the fluid, and b the height of the quickſilver, which ballances the preſſure of the atmoſphere, then will $b-x$ expreſs the height of the quickſilver in the tube, and $a+b-x$ that of the quickſilver and fluid together.

Now if m is to n as the ſpecific gravity of the quickſilver is to that of the fluid; then becauſe theſe columns have the ſame baſe, they are as
Part. 571. their altitudes; and ſince the weights are as * the ſpecific gravities and magnitudes conjointly, an will be the weight of the fluid, and mx that of the quickſilver, which is equal to it; and therefore $mx=an$ by ſuppoſition, or $x=\frac{an}{m}$. This value being ſubſtituted into $b-x$, and $b+a-x$, gives $b-\frac{na}{m}$, for the altitude of the quickſilver, and $a+b-\frac{na}{m}$, for that of the quickſilver and fluid together.

Examp. Let there be water in the tube to the height of 28 inches, and the height of the quickſilver in the barometer 30 at the ſame time; then becauſe m is to n as 14 is to unity, according to the table of ſpecific gravity, we have $b=30$, $a=28$, $m=14$, $n=1$, $b-\frac{na}{m}=28$, for the height of the quickſilver, and $a+b-\frac{na}{m}=56$, for that of the water and quickſilver together.

SECT.

SECT. VIII.

Of PUMPS.

HAVING applied the principles laid down in every ſection to ſome uſeful ſubject, we cannot but chuſe to do the ſame here, in regard to hydrolics; but becauſe of its great extent it is impoſſible to enter into all the particulars in ſuch a work as this, we ſhall content ourſelves to explain the nature and uſe of pumps only, as being the moſt ſimple, and at the ſame time the moſt uſeful, of all hydrolic machines.

There are three ſorts, the ſucking, the lifting, and the forcing pump; we ſhall give a deſcription of each ſeperately, after having ſaid ſomething concerning the ſeveral parts of which they are compoſed.

The body A B C D of the pump is called the *Barrel*, which is a hollow cylinder, made of braſs, copper, potmetal, a compoſition of lead and copper, hard metal made of lead and tin, or hard lead; that is, the barrel is made of any ſubſtance which may be made ſmooth within; the wooden ones are made only for cheapneſs. *Fig.* 30.

The ſucking pipe E A D F, is that which goes from the barrel down into the water, which is generally widen'd at the lower end, to give a free paſſage for the water to enter; there is fixed a thin iron plate at the entrance, full of holes, to prevent the dirt from getting into the pipe. This is

is made of any kind of hard or ſoft ſubſtance, as wood, lead, or caſt iron; becauſe it requires not to be ſo ſmooth within as the barrel. Theſe two cylinders are joined together at A D by two flat rings caſt with them, called by the *French* Bridles, ſcrewed together with four ſcrews, and to prevent the air or water from paſſing between, a ring of leather is put into the joint, which is here ſtained darker than the pipes: But when the pump is made of wood, the barrel and ſucking pipe are both made of the ſame piece, or the ſucking pipe is leſs than the barrel, and its end ſharpened ſo as to be forced into the barrel, which is bound with an iron hoop to prevent its ſplitting. Common pumps are often made of a ſingle leaden pipe.

The *Piſton* is a ſolid *a b c*, which moves in the barrel by means of a power applied to an iron rod *e f*, fixed to the handle *d d*; there are two ſorts of piſtons, the one hollow within, ſuch as are repreſented in *fig.* 30 and 31; where the cavity is marked with two pricked lines which is called *Bucket*, of which there are ſeveral kinds; the moſt ſimple of all, and which is commonly uſed in ordinary pumps, is made of wood, in the form of a trunkated cone, with its leſſer baſe downwards, having an iron hoop or ring at each end, to prevent the wood from ſplitting; and about the upper part *b c* is faſtened a piece of ſtrong leather, reaching a little above the wood, by two rows of nails, in order to keep the bucket tight and cloſe to the barrel, ſo as no water or air may paſs between them.

The cavity of the bucket is covered by a *Valve g*, which is a round piece of leather, exceeding the

the diameter of the hole about an inch, and having a tong or ſlipe on one ſide, which being nailed on the bucket ſerves as a joint for the valve to turn about; and to prevent the leather from bending, a thin plate of iron is put over it of the the ſame diameter, and ſometimes another under it, of the bigneſs of the cavity of the bucket, all three rivetted together, or faſtened with ſcrews.

In ſucking pumps, where the weight of the water above the bucket is not very great, a piece of lead is only faſtened to the leather by a ſcrew, ſuch as is repreſented here at *g*, to make the valve ſhut by its own weight. Theſe ſort of buckets will be ſufficient in ordinary pumps, being beſides the cheapeſt of all; but in larger ones the beſt way of making them is as follows.

Inſtead of the wooden cylinder, or rather trun- *Fig.* 33.
cated cone, let there be a braſs one *m n*, with a brim or projection k *x* l at the upper end of about half an inch broad, and ſo as to project the lower part by the thickneſs of ſtrong leather, the part *x* m below this brim is covered with ſtrong leather, faſtened with an iron hoop m ſunk into the leather ſo as not to project it.

There is a braſs bar at each end, caſt with the bucket, ſuch as is ſeen at the upper end k l, with a ſquare hole in the middle to receive the iron rod or handle of the piſton. The valve of this bucket is a round piece of leather, nearly equal in diameter to that of the bucket, on each ſide of which are faſtened two equal iron plates, with rivets or ſcrews in the form of a ſegment of a circle, ſo as to leave the breadth of the bar k l of the leather uncovered, which ſerves as a joint to both parts to turn about; this part of the leather uncovered is laid on the bar k l, and faſtened cloſe to it by the

the iron piece r s, *fig.* 34, ſo that one end t being placed at k, the other *v* will be at l; the lower end r of this iron goes through the ſquare holes of both bars, and is faſtened underneath with a key, and to the upper end s is fixed the iron rod by which the piſton is moved.

The two ſegments of the valve riſe together by the preſſure of the water coming through the bucket, and lean againſt the iron r s; and when the bucket aſcends, fall back again and ſhut the cavity of the bucket.

Fig. 32. The ſolid piſton *a b c* uſed in forcing pumps, is a ſolid cylinder of wood or braſs, ſomewhat leſs in the middle than at the ends, to receive a leathern collar, which is nailed on it, when the cylinder is of wood, ſuch as is repreſented here; ſo that the whole together keeps the piſton tight and cloſe to the barrel, that no air or water may paſs between them: The iron rod by which the piſton is moved paſſes through it, and is faſtened by two keys, one below and the other above it.

Theſe ſort of piſtons are ſometimes made of three braſs cylinders which ſcrew into each other; the middle one fills the barrel of the pump nearly, and the others are ſomewhat leſs; and to make the piſton come cloſe to the barrel, two leathern rings are inſerted into the joints, by which no air or water can get between the piſton and the barrel.

Authors have imagined various ſorts of piſtons, as well as different valves; but the moſt material ones, and which are commonly uſed, have been deſcribed; as for the reſt, we ſhall refer the reader to Mr. *Belidor*'s Hydrolics, or to Dr. *Deſagulier*'s Experimental Philoſophy.

Mr. *Belidor* propoſes a new ſort of valves, which is a braſs circular plate, with an axis fixed

to

to it, ſo as to divide the area into two ſegments, one being double the other; this plate is made very ſmooth, and ground ſo true to the bucket as to let paſs no water or air between them.

When the bucket deſcends and the water preſſes againſt this valve, it finds leſs reſiſtance on one ſide than on the other, by which the valve is forced to turn about its axis, with the larger ſegment upwards, to let paſs the water; and when the piſton aſcends, the weight of the air or water above it preſſes more on the larger ſegment than on the ſmaller, by which the valve is ſhut again.

In the lifting pump the piſton is the ſame as in *Fig.* 31. the ſucking one, only inverted, and the valve is at the ſmaller end of the bucket.

At the end of the barrel where it is joined to the ſucking, lifting, or forcing pipe, there is likewiſe a valve *z*, made in the ſame manner as thoſe in the piſton, for which reaſon we ſhall not enter into any particulars about it, obſerving only that the leathern tong of the valve is of a piece with the leathern ring between the joints of the pipes. When the barrel and ſucking pipe are of a piece, the valve is fixed near the ſurface of the water, and ſometimes under it; the box to which the valve is fixed is either a wooden or braſs cylinder, and ſoldered to the pipe if it be of any kind of metal, and forced down when made of wood; in this latter caſe it will be convenient to have an iron ring fixed to it, in order to draw it out when occaſion requires it.

Having explained the ſeveral parts of which pumps are compoſed, we ſhall now proceed to ſhew the manner of working them.

Of

Of the SUCKING PUMP.

Fig. 30. When the piſton deſcends to AD, the air contained between it and the valve *z*, being reduced
* *art.* 616. into a leſſer ſpace, will be condenſed *, and by its elaſtic force preſſes the valve *z* down, and forces the valve *g* of the bucket open, ſo as to riſe above the piſton; then by riſing the piſton the weight of the atmoſphere preſſes the valve *g* down, ſo as no air can paſs through the bucket, by which the air that remained between the bucket and the valve *z*, will be rarified; and as that in the ſucking pipe is denſer than that above this valve, it acquires an elaſtic force, ſufficient to raiſe the valve *z*, and ſo enters into the barrel, till the air in both is of the ſame denſity; then the atmoſphere preſſes on the ſurface of the water in the well, cauſes the water to riſe in the pipe; and when the piſton deſcends, the air under the piſton keeps the valve *z* ſhut, forces the valve *g* of the bucket open, and paſſes above the piſton; and by riſing the piſton, the preſſure of the ambient air keeps the valve *g* ſhut, and the air in the ſucking pipe forces the valve *z* open, enters into the barrel, and the water in the well being preſſed by the atmoſphere riſes again in the pipe: by continuing in this manner to move the piſton, the water will riſe into the barrel, and from thence into the cyſtern or reſervoir KL, provided the height of the cyſtern above the ſurface of the water in the well does not exceed 34 feet, or
* *art.* 626. thereabouts: For we have ſhewn * that a column of quickſilver of 29.5 inches high, was in equilibrio, with the preſſure of the atmoſphere, and that water is about 14 times lighter than quickſilver;

ſilver; therefore a column of water whoſe height is $14 \times 29.5 = 413$ inches, or 34 feet and 5 inches, will counterpoiſe the weight of the atmoſphere on the ſame baſe.

PROB.

633. *To find the height to which water will riſe in the ſucking pipe at every ſtroke of the piſton, the length of the ſucking pipe and that of the ſtroke being given.*

Suppoſe the piſton to deſcend quite cloſe to the valve z, divide the capacity of the barrel from AD to the end of the ſtroke, by the ſection of the ſucking pipe, in order to reduce them both to the ſame diameter; and let c expreſs the height of the column of water which is in equilibrio with the preſſure of the atmoſphere, a the length of the ſtroke of the piſton, b the length of the ſucking pipe, and x the height to which the water riſes in the pipe at the firſt ſtroke.

When the piſton is raiſed, the air in the pipe would be rarified into the ſpace $a+b$, if the water did not riſe in the pipe; but the water riſing to the height x, the air will be reduced into the ſpace $a+b-x$: Therefore its denſity before the ſtroke, will be to its denſity after the ſtroke *, reciprocally as the ſpace $a+b-x$ is to the ſpace b; but becauſe the air was in equilibrio with the atmoſphere before the ſtroke, it will be of the ſame denſity, and ſo its elaſtic force may be expreſſed by c: Whence, if $a+b-x : b :: c : \frac{bc}{a+b-x}$, this fourth term will expreſs the denſity or elaſtic force of the air after the ſtroke; which, together with the weight of the water x, are in equilibrio with the preſſure of the atmoſphere im-

* art 617.

immediately after the ſtroke. Conſequently, $\frac{bc}{a+b-x}+x=c$, or $\frac{bc}{a+b-x}=c-x$, and $bc=ac+bc-ax-bx-cx+xx$, by multiplication. If $a+b+c=2n$, we ſhall have $xx-2nx=-ac$, by reduction; and adding nn to both ſides, the firſt will be a compleat ſquare, whoſe root is $n-x=\sqrt{nn-ac}$, or $x=n-\sqrt{nn-ac}$.

The height to which water riſes in any of the ſucceeding ſtrokes will be found by ſubſtituting the difference between the height of the ſucking pipe and that of the water in it, inſtead of b into the equation above.

COR.

634. Hence, if it be required to find the length of the ſucking pipe, ſo as the water ſhall riſe to any given height x; the equation $ac-ax-bx-cx+xx=0$, gives $b=x+\frac{ac}{c+x}$; and if the length of a ſtroke was required to make the water riſe to the given height x; then will $a=\frac{bc}{x}+x-c-b$.

EXAMPLE.

Let $c=31$ feet, the height of the ſucking pipe $b=28$, and the length of the ſtroke $a=8$; then will $a+b+c=2n=67$, or $n=33.5$; and ſubſtituting the values of a, b, c, n, into the equation $x=n-\sqrt{nn-ac}$, we ſhall have $x=33.5-29.5=4$ feet nearly, for the height to which the water riſes at the firſt ſtroke; and the ſpace b to which the air is then reduced into, will be $28-4=24$ feet.

Now

Now if we ſuppoſe $b = 24$, and the reſt as before, then will $a + b + c = 2n = 63$, or $n = 31.5$, and $x = 31.5 - 27.3 = 4.2$, for the height to which the water riſes at the ſecond ſtroke; and the air will be contained in the ſpace of $24 - 4.2 = 19.8$. Whence, if $b = 19.8$, then $a + b + c = 2n = 58.8$, or $n = 29.4$, and $x = 29.4 - 24.8 = 4.6$, for that height at the third ſtroke. In the ſame manner the height at the fourth ſtroke is found to be 5.1, and 8.7 at the fifth; when the air is reduced into a ſpace of 1.4 feet, which being leſs than the height of the ſtroke, the water muſt riſe above the piſton at the ſixth, and from thence into the cyſtern.

It muſt be obſerved that this does not anſwer exactly in practice, becauſe part of the air which enters into the barrel is forced down again into the ſucking pipe by the ſhutting of the valve; but, however, the difference is not very conſiderable.

Theſe are the dimenſions of a pump, in which Mr. *Belidor* ſays that the water will never riſe * to a height of 10 feet; and he is ſurprized that Mr. *Mariotte* ſhould propoſe it as a pattern for this kind of pumps: But I am more ſo, that he ſhould not perceive the abſurdity of this argument, and that Dr. *Deſagulier* ſhould fall into * the ſame miſtake.

*Page 90.

*Page 183

In order to prove this he refers to *art.* 815, where he ſays, as the air cannot be pumped out to that degree ſo as none remains in the pump, the water will not quite riſe 32 feet high; which he takes to be the height of the column of water that counterpoizes the preſſure of the atmoſphere: But how this may account for the water not riſing above 9 feet and ſome inches, I leave the reader to judge.

It would be needleſs to expatiate any further upon ſuch an abſurd argument, and therefore we ſhall proceed in our ſubject.

Of the LIFTING PUMP.

Fig. 31. The barrel ABCD of this pump is ſuppoſed to be in the water; to the lower end AD is ſcrewed a tube ADHG, wider at bottom than the barrel, to let the water run freely into the pump; the barrel is often made wider itſelf at the end, which ſaves expences; BEFC is a part of the lifting pipe which goes up to the cyſtern or reſervoir. When the piſton is raiſed the water or air above it keeps the valve *g* ſhut, and forces the valve *z* open to enter into the lifting pipe; then when the piſton deſcends, the water or air above the valve *z* preſſes it down and keeps it ſhut, whilſt the water in the well forces open the valve *g* and enters above the piſton, which, by riſing, lifts it up into the lifting pipe as before.

It is plain that this pump will riſe water to any height, provided the moving power be ſufficient. This power lifts up not only the water above the piſton, but likewiſe that in the barrel below it; becauſe when it riſes it cauſes a vacuum between the ſurface of the water and the bucket, till the preſſure of the outward air forces the water to riſe in the pump; and therefore, as much of the preſſure of the air is taken off in the tube, as is equivalent to the weight of the water which riſes.

Of the FORCING PUMP.

Fig. 32. The piſton *abc* of this pump is a ſolid cylinder, as we have obſerved before; the forcing pipe

pipe GHID is at firſt perpendicular to the barrel, and then it riſes gradually up to the cyſtern or reſervoir, or it turns up vertically, according as the cyſtern is near or at a diſtance from the well.

When the piſton deſcends it compreſſes the air under it in the barrel, by which it paſſes into the forcing pipe, through the valve *g*; then when the piſton riſes it rarifies the air above the valve *z*, by which the air or water under it forces it open to enter into the barrel, whilſt the valve *g* is kept ſhut by the preſſure of the air in the forcing pipe; and by forcing the piſton downwards, the air above the valve *z* is forced out through the valve *g*; and thus by continuing to move the piſton, the water will riſe into the barrel by the preſſure of the atmoſphere, and from thence it is forced up the forcing pipe into the reſervoir.

This pump ſerves to raiſe water to any height as well as the former, and is preferable to it, inaſmuch as the weight of the forcer diminiſhes the moving force; and it may be charged with ſuch a weight, as that the power of forcing up the water be equal to that which raiſes the forcer; by which the motion of the machine becomes uniform, which is a great advantage, and cannot be had in lifting pumps, where the moving power is obliged to lift up the weight of the piſton and the tackle belonging to it, beſides the water.

The MANNER *of applying the* MOVING POWER *to the* PISTON.

In the ſucking pump the power is applied to a ſhort leaver which turns about an axis, the end of which, where the iron rod *e f* is faſtened to, is *Fig.* 30.

but ſhort, and ſeldom exceeds 6 inches; ſome times an iron bar is fixed to one end of the axis, with a weight at the end, and the leaver where the rod of the piſton is faſtened to, is perpendicular to the middle of this axis, as may be ſeen in moſt common pumps.

If it ſhould be required to make the ſtroke of the piſton pretty long, it would be proper to add a circular arc to the leaver, then the iron rod *e f* is faſtened to it, by means of a chain of a few links to roll upon that arc, as may be ſeen in the fire-engine at *Chelſea*: But then, the piſton muſt be heavy enough to deſcend by its own weight. As in common pumps the ſtroke of the piſton is moſtly very ſhort, this method is ſeldom practiſed.

Fig. 31. In the lifting pump the iron rod of the piſton is fixed to a rectangular iron frame, not unlike the wooden frames of ſome ſaws; and this frame is fixed with another iron rod to the cranks of an axis of a wheel, which is turned by water or horſes; and to looſe no power, the axis has generally two cranks bending contrary ways, in order to move two piſtons, ſo that when one aſcends, the other deſcends: The barrels of theſe two piſtons are both joined to one lifting pipe, which therefore carries the water of both into the reſervoir by a continued ſtream.

Fig. 32. In forcing pumps the iron rod of the forcer is fixed to a crank of an axis, in the ſame manner as in a lifting pump. There are various manners of applying the moving powers to the piſton, that are uſed upon different occaſions, which are impoſſible to be all deſcribed here, and may beſt be ſeen in authors who have wrote particularly upon this ſubject.

There

There is ſometimes an air pipe added to the lifting or forcing pump, which is upright, and has a communication with the lifting or forcing pipe, of the ſize, or bigger than the barrel of the pump, and ſtopped cloſe above with a cover ſcrewed on it. When the air or water is forced into the lifting or forcing pipe, the air is condenſed in the air pipe, and part of the water enters into it; and when the piſton works, this air forces the water to riſe in the pipe, by which it runs into the reſervoir with a continued uniform motion; which is a great advantage to the machine, although it produces no more water than if there was none; but it muſt be obſerved, that this pipe is only uſed in a ſingle pump; for when there are more than one, there is no occaſion for it.

PROPORTIONS *of the ſeveral parts of a* PUMP.

As the goodneſs of all machines depends on the exactneſs of the proportions between the parts, in reſpect to each other, we ſhall endeavour to explain here, in a few words, all that can be ſaid of pumps: in order to which we will begin with thoſe between the diameter of the barrel and the ſucking pipe, which ought to be ſuch, that the diameter of the cavity in the bucket, is exactly equal to that of the ſucking pipe. For when this is the caſe, the water will riſe every where with an equal velocity, which is allowed to be a great perfection; ſince if ſome parts are narrower than others, the velocity will there be ſo much the greater in proportion; and as the reſiſtance of the pipe partly depends on the velocity, it muſt be *Fig.* 30.

made proportionably ſtronger there than in any other part: If the barrel be wider, the moving power muſt be greater; ſince the preſſure of the air is always proportional to the ſurface preſſed; that is, as the baſe of the piſton.

The ſame thing muſt be obſerved with reſpect to the valve z; for wherever it be placed, its opening ſhould always be the ſame as that of the bucket.

Mr. *Belidor* would have the valve z placed at the end of the barrel, in the manner repreſented in *fig.* 30; and Dr. *Deſagulier* at the end of the ſucking pipe, near or under the ſurface of the water. I muſt confeſs, that I cannot perceive any advantage in placing it in one part ſooner than in another; for when it is at the end of the barrel, it may be eaſier come at, in caſe it wants mending, which it will oftener than if it was under water, where it is conſtantly kept wet, whereby it will be but ſeldom out of order: As to the goodneſs of the pump, it does not in the leaſt depend on the ſituation of this valve, ſince it will diſcharge as much water one way as the other, at the ſame time.

The next thing to be conſidered is, the velocity with which the piſton ought to move, in reſpect to that with which the water riſes in the pipe; for if the piſton moves faſter than the water, there will be a vacancy left between them; and when the piſton deſcends, it will not fetch ſo much water, as it ſhould do; and if the velocity of the water is greater than that of the piſton, it will be ſtopp'd; and therefore the pump will not furniſh the quantity of water which it might, if the piſton did move faſter; for which reaſon we ſhall give the following

PROB.

PROB. II.

To find the velocity of the water in the ſucking pipe, at any given height.

As the water riſes in the pipe by the preſſure of the atmoſphere on the ſurface of the water in the well, and this preſſure is equivalent to the weight of a column of water of 34 feet height, as has been obſerved before; it is evident that this preſſure is capable to generate a velocity equal to that of a body falling through that height; ſince, if the air is removed out of the pipe, the water would riſe to that height; therefore the velocity of the water, at any height from the ſurface of the water in the well, is equal to the velocity of a body thrown upwards with a velocity acquired by falling through 34 feet, ſetting aſide all obſtacles or impediments; but the velocity of the body at any height, when thrown upwards, is equal to that which a body would acquire by * falling through the difference, between the aſcent and deſcent. If therefore a expreſſes the difference between the height of the pump and 34 feet; then becauſe a body falls thro a ſpace of 16.1 feet in a ſecond, and with the velocity acquired during the fall, it would deſcribe a ſpace 32.2 double, uniformly in the ſame time; and ſince the velocities acquired are as the ſquare roots of the heights; if $\sqrt{16.1} : 32.2 :: \sqrt{a} : 2\sqrt{16.1a}$, this fourth term will be the ſpace deſcribed uniformly with the velocity of the water in a ſecond of time. * art. 453.

Examp. Let the height of the pump from the ſurface of the water in the well to the cyſtern be 26 feet; then will $34 - 26 = 8 = a$; and therefore $2\sqrt{16.1a}$, gives 11.3 feet for the ſpace deſcribed

ed uniformly with the velocity of water in a second of time.

As the water meets with some resistance in the ascent of the tube, this velocity is rather more than the water really has at that height; for which reason the velocity of the piston should not be so much.

Mr. *Belidor* says, that he has seen no pump whose piston had above 4 *French* feet velocity in a second; if it be so, there is no danger that the water will not follow immediately the piston, at least when the pump is not higher than about 26 feet.

If the cavity of the bucket be greater than that of the valve z, or the section of the sucking pipe; the water will move slower in the barrel, in a reciprocal proportion of the section of the sucking pipe to that of the cavity of the bucket: Whence, if D be the diameter of the sucking pipe, d that of the cavity in the bucket, by making $dd : DD :: 2\sqrt{16,1a} : x$; the fourth term x will express the velocity of the piston such, that the water will immediately follow the piston.

Fig. 31. In the lifting pump the piston should move quite close up to the valve z, because the water which is not lifted through that valve, descends again with the piston, is an additional weight to be lifted up by the moving power, and the pump does not discharge the quantity of water it ought to do.

Fig. 32. As to the piston in the forcing pipe, it ought not descend lower than to about half the diameter GD of the forcing pipe; the valve z should be quite even with the bottom of that pipe; since whatever water remains above that valve is a weight to it, and prevents the water from rising

so

ſo quick as it ſhould do. And if the forcing pipe did incline a little downwards from the barrel to the valve *g*, ſo as the water which is not forced out, might run through that valve by its own weight, the pump would, in my opinion, be better than when that pipe makes a right angle with the barrel; ſince by that means all the water that riſes above the valve *z*, would be diſcharged without any loſs.

Mr. *Belidor* has given eight problems * relating to the perfection of pumps, formerly propoſed by Mr. *Parent*, which were taken out of a manuſcript containing a treatiſe on pumps, he wonders how it came not to be printed, and imagines that Mr. *Parent* wanted to make a ſecret of it. As I have conſidered with great attention the ſolutions given of them by Mr. *Belidor*, I cannot find that they are of any uſe, to the beſt of my judgment, which makes me believe that Mr. *Parent* found himſelf miſtaken in propoſing of them, ſuppreſſed the manuſcript from which they were taken; but Mr. *Belidor* being fully bent upon giving, in his works, whatever ſeemed to be conducive to the perfection of machines, relying too much on Mr. *Parent*'s skill and knowledge, has without duly conſidering whether it was right or not, publiſhed theſe problems, which by the author himſelf were found to be of no conſequence.

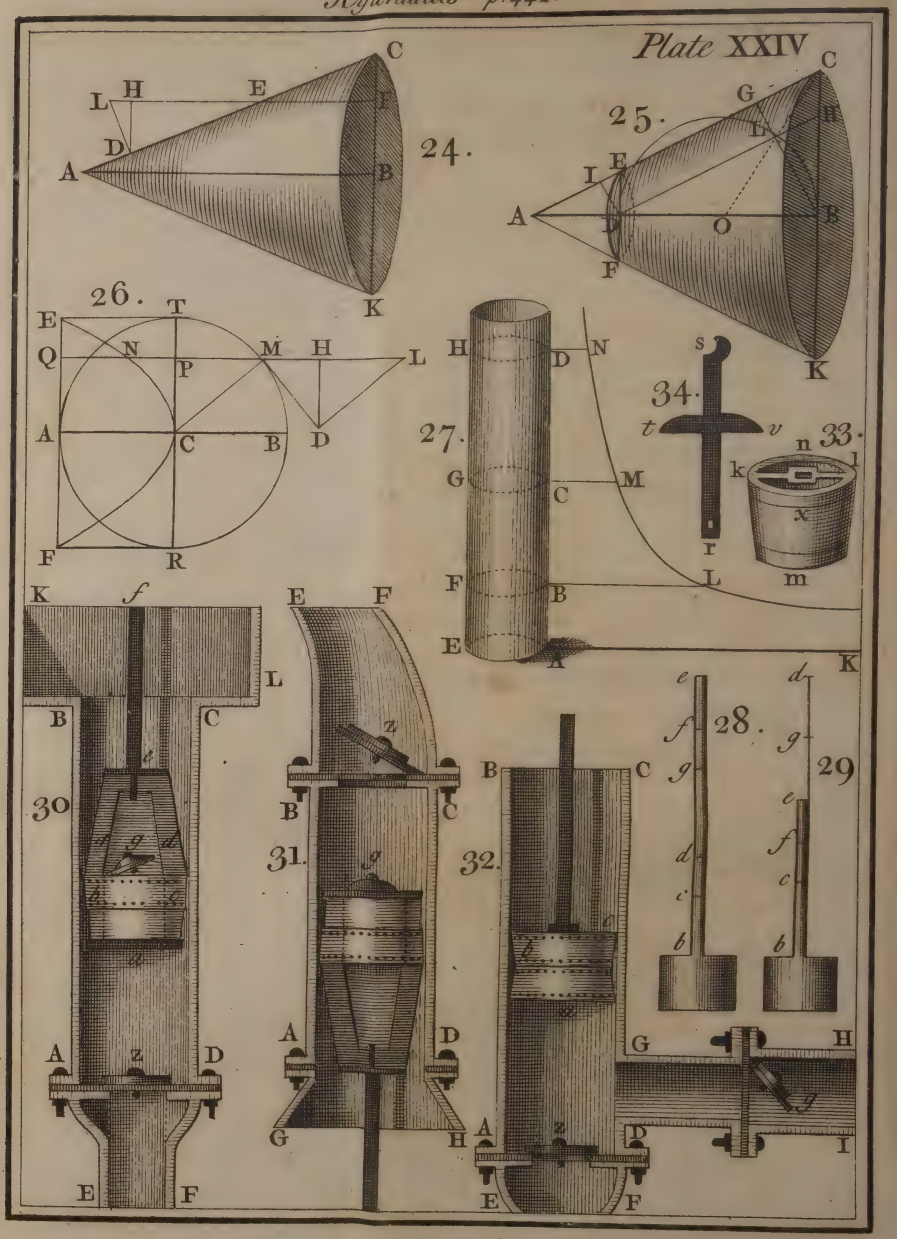
* *Pag.* 92.

This is nearly all that can be ſaid in regard to the pumps we have here been ſpeaking of; there are, it is true, abundance of different manners in which they are adapted to machines, by which they are worked; but as our deſign in this work is only to treat of ſuch parts as are moſt uſeful to an engineer, or any other military gentleman, thoſe whoſe curioſity incites them to ſtudy any par-

part, in a more compleat manner, muſt conſult thoſe authors who wrote particularly on that ſubject; the beſt of each kind ſhall be inſerted at the beginning of this work.

P. S. In peruſing the memoirs of the royal academy of ſciences at *Paris*, of the year 1735, I found a treatiſe on pumps, wrote by Mr. *Pitot*, wherein he endeavours to prove, that in many pumps the water will not riſe up to the cyſtern, though it be not 32 feet high; and in an example where he ſuppoſes the ſtroke of the piſton 2 feet, and the ſucking pipe 14 long, that the water will not riſe above 8 feet; as we have already taken notice of this miſtake in page 433, we ſhall only add, that both Mr. *Belidor* and *Pitot* take the exhauſting of the air out of an air pump and that out of a water pump to be the ſame thing; they do not conſider, that when the air is exhauſted out of the firſt, there is nothing that ſupplies its place; whereas in the latter, the water riſes as the air is rarified, till it is again of the ſame denſity with the outward air; and therefore no reaſon can be aſſigned, why the water ſhould riſe no higher than theſe gentlemen would have it: If any ſuch thing happens in practice, as they ſeem to inſinuate, it muſt be owing to ſome imperfection of the pump, and not to what they have imagined. Mr. *Pitot* gives likewiſe the ſolutions of Mr. *Parent*'s 8 problems; but as his principles are not better grounded than thoſe given by Mr. *Belidor*, I have no reaſon to contradict what has been ſaid in page 441, on that account.

F I N I S.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